ON THE THEORY OF GASEOUS JETS

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This paper contains the solution and analysis of a number of problems in the theory of gaseous jets. The first section discusses the question of a converging gaseous jet; the second considers flow past a plate under the assumption that in front of the plate there exists a region of gas at zero velocity. In spite of their apparent diversity, these problems have a common property, namely, that the first problem is obtained from the well known problem of the jet flow of a gas from an infinite vessel by means of substituting the point of zero velocity at infinity by a whole unbounded region of stagnation; a similar stagnation region in the second problem is considered to be in front of the plate.

The third section contains the solutions of a number of problems on the jet flow of a gas which were studied earlier by Zhukovskii for the case of an incompressible fluid. For the solutions of these problems we make use of new particular integrals of Chaplygin's equation

I. The problem of a contracting gaseous jet

1. Let us consider a stream of gas, flowing with velocity V_1; let us assume that the width of this stream, and also the density of the gas, are known. Let us assume further that, in its motion, the gas encounters two straight walls, symmetrically disposed relative to its direction of motion, and including between them an angle 2λ, and that a jet with free surfaces issues from the orifice formed by these two walls. Our problem consists in determining the entire motion of the gas by means of the methods described by Chaplygin [1] in his paper *On gaseous jets*. In what follows we adopt the notation of that paper for all the principal quantities.

Let us assume that the gas flowing from infinity has a velocity parallel to the positive direction of the x-axis; this axis is the line of symmetry of the two guiding walls, and the origin of coordinates is taken at the point of intersection of this axis with the line joining the ends of the walls, from which the gas jet emerges. Let us denote by V_1 the constant velocity of the stream at infinity before its encounter with the walls. This is also the velocity of the gas particles on those streamlines along which these particles move before striking the walls.* Let us, moreover, denote by V_2 the velocity of the gas particles along the two streamlines issuing from the inclined walls and bounding the jet directed along the positive axis of x.

Adopting Chaplygin's notation, we set

$$\tau_1 = \frac{V_1^2}{2a}, \qquad \tau_2 = \frac{V_2^2}{2a}$$

Let us denote by ρ_1 the density of the incident gas, and by ρ_2 the density of the gas in the remote parts of the emergent jet. If we call the width of the incident stream $2\,l_1$, and the width of the emergent jet at infinity $2\,l_2$, we shall then have

$$\rho_1 l_1 V_1 = \rho_2 l_2 V_2 \tag{1.1}$$

From Bernoulli's equation, written in the form

$$\rho = \rho_0 (1 - \tau)^{\beta}$$

we obtain the two relations:

$$\rho = \rho_1 \left(\frac{1-\tau}{1-\tau_1}\right)^{\beta}, \qquad \frac{\rho_2}{\rho_1} = \left(\frac{1-\tau_2}{1-\tau_1}\right)^{\beta} \tag{1.2}$$

If the axis of x delineates the zero value of the stream function ψ, then along the streamline $A'B'C'D'$ the function ψ is equal to a constant value $q = \rho_1 l_1 V_1/\rho_0 > 0$, and along the streamline $ABCD$ it is equal to the constant value $-q$ (Fig. 1).

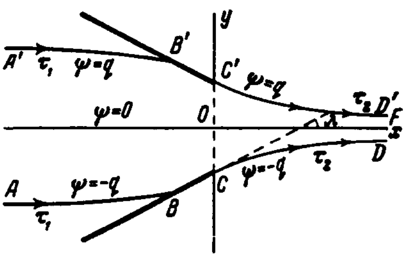

Fig. 1.

The function ψ, considered as a function of τ and the angle of inclination θ of the gas particle velocity to the axis of x, satisfies the

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* *Translator's Note:* This refers to the two free streamlines $A'B'$ and AB inside the vessel. These free streamlines separate the moving stream from the stagnant fluid which fills the remainder of the vessel.

following equation:

$$\frac{\partial}{\partial\tau}\left\{\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}\right\}+\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2}=0 \tag{1.3}$$

We find the integral of this equation subject to the relevant boundary conditions for those values of θ and τ which correspond to the lower half of the stream between $ABCD$ and the axis of x; here the angle θ varies from zero to λ, and the variable τ varies from τ_1 to τ_2. The boundary conditions for the determination of the function $\psi(\theta, \tau)$ are written thus (Fig. 2):

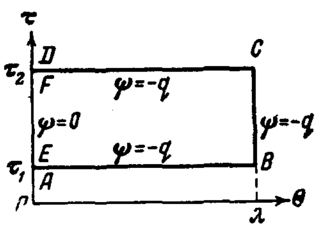

Fig. 2.

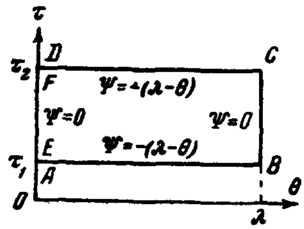

Fig. 3.

$$\begin{aligned}
&\text{when } \tau=\tau_1 \text{ and for } 0<\theta<\lambda \text{ the function } &&\psi=-q\\
&\text{when } \theta=\lambda \text{ and for } \tau_1<\tau<\tau_2 \text{ the function } &&\psi=-q\\
&\text{when } \tau=\tau_2 \text{ and for } \lambda>\theta>0 \text{ the function } &&\psi=-q\\
&\text{when } \theta=0 \text{ and for } \tau_2>\tau>\tau_1 \text{ the function } &&\psi=0
\end{aligned} \tag{1.4}$$

Together with the function $\psi(\theta, \tau)$ we will determine the function $\Psi(\theta, \tau)$, the integral of equation (1.3), connected with the function $\psi(\theta, \tau)$ by the relation

$$\psi=\frac{q}{\lambda}(\Psi-\theta) \tag{1.5}$$

The new function Ψ must satisfy the following boundary conditions (Fig. 3):

$$\begin{aligned}
&\text{when } \tau=\tau_1 \text{ and for } 0<\theta<\lambda \text{ the function } &&\Psi=-(\lambda-\theta)\\
&\text{when } \theta=\lambda \text{ and for } \tau_1<\tau<\tau_2 \text{ the function } &&\Psi=0\\
&\text{when } \tau=\tau_2 \text{ and for } \lambda>\theta>0 \text{ the function } &&\Psi=-(\lambda-\theta)\\
&\text{when } \theta=0 \text{ and for } \tau_2>\tau>\tau_1 \text{ the function } &&\Psi=0
\end{aligned}$$

To determine the function Ψ we find particular solutions of equation (1.3). This equation has a particular solution of the following form:

$$\Psi_n(\theta,\tau)=z_n(\tau)\sin\frac{\pi n}{\lambda}\theta \tag{1.6}$$

where n is any integer greater than or equal to unity. The function $\Psi_n(\theta,\tau)$ evidently satisfies the condition

$$\Psi_n(0,\tau)=\Psi_n(\lambda,\tau)=0$$

The function $z_n(\tau)$ is the general integral of the equation

$$\frac{d}{d\tau}\left\{\frac{2\tau}{(1-\tau)^\beta}\frac{dz_n}{d\tau}\right\}-\frac{\pi^2n^2}{\lambda^2}\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}z_n=0 \qquad (1.7)$$

Let us now construct a series of particular solutions of equation (1.7). Let us, however, first introduce the following notation: we will denote by $z_{n1}(\tau)$ the integral of equation (1.7) satisfying the conditions

$$z_{n1}(\tau_1)=0, \qquad \frac{dz_{n1}(\tau_1)}{d\tau}=1$$

and by $z_{n2}(\tau)$ we will denote the integral of equation (1.7) satisfying the conditions

$$z_{n2}(\tau_2)=0, \qquad \frac{dz_{n2}(\tau_2)}{d\tau}=1$$

With this notation we can write

$$z_n(\tau)=C_{n1}z_{n1}(\tau)+C_{n2}z_{n2}(\tau)$$

where C_{n1} and C_{n2} are two arbitrary constants.

Let us now expand the function $\Psi(\theta, \tau)$ in the series

$$\Psi(\theta,\tau)=\sum_{n=1}^{\infty}[C_{n1}z_{n1}(\tau)+C_{n2}z_{n2}(\tau)]\sin\frac{\pi n}{\lambda}\theta$$

and determine the constants C_{n1}, C_{n2} from the following boundary conditions:

$$\Psi(\theta,\tau_1)=-(\lambda-\theta) \quad \text{for } 0<\theta<\lambda, \qquad \Psi(\theta,\tau_2)=-(\lambda-\theta) \quad \text{for } 0<\theta<\lambda$$

Applying the theory of Fourier series, we obtain

$$C_{n2}=-\frac{2\lambda}{\pi n}\frac{1}{z_{n2}(\tau_1)}, \qquad C_{n1}=-\frac{2\lambda}{\pi n}\frac{1}{z_{n1}(\tau_2)}$$

whence we find that

$$\Psi(\theta,\tau)=-\frac{2\lambda}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{z_{n1}(\tau)}{z_{n2}(\tau_2)}+\frac{z_{n2}(\tau)}{z_{n2}(\tau_1)}\right]\sin\frac{\pi n}{\lambda}\theta$$

Returning to formula (1.5), we find the stream function:

$$\psi=-q\left\{\frac{\theta}{\lambda}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{z_{n1}(\tau)}{z_{n1}(\tau_2)}+\frac{z_{n2}(\tau)}{z_{n2}(\tau_1)}\right]\sin\frac{\pi n}{\lambda}\theta\right\} \qquad (1.8)$$

Using the formulas

$$\frac{\partial\varphi}{\partial\theta}=\frac{2\tau}{(1-\tau)^\beta}\frac{\partial\psi}{\partial\tau}, \qquad \frac{\partial\varphi}{\partial\tau}=-\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta}$$

we then find an expression for the velocity potential:

$$\frac{\lambda\varphi}{q} = \int_{\tau_1}^{\tau} \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\,d\tau + \frac{4\lambda^2}{\pi^2}\frac{\tau}{(1-\tau)^\beta}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\frac{z_{n1}'(\tau)}{z_{n1}(\tau_2)} + \frac{z_{n2}'(\tau)}{z_{n2}(\tau_1)}\right]\cos\frac{\pi n}{\lambda}\theta \qquad (1.9)$$

2. With the help of formulas (1.8) and (1.9), let us find a number of geometrical quantities relating to the dimensions and shape of the stream.

First of all, let us calculate the distance b of the point B or B' from the axis of the stream.

We have the general formula:

$$dy = \frac{\partial y}{\partial \varphi}\,d\varphi + \frac{\partial y}{\partial \psi}\,d\psi$$

Applying this to the streamline $\Psi = -q$ from the point A to the point B, we obtain

$$dy = \frac{\partial y}{\partial \varphi}\frac{\partial \varphi}{\partial \theta}\,d\theta = -\frac{\sin\theta}{\sqrt{2\alpha\tau_1}}\frac{4q}{\pi}\frac{\tau_1}{(1-\tau_1)^\beta}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{z_{n1}'(\tau_1)}{z_{n1}(\tau_2)} + \frac{z_{n2}'(\tau_1)}{z_{n2}(\tau_1)}\right]\sin\frac{\pi n}{\lambda}\theta\,d\theta$$

Let us integrate this equation from the point A to the point B; taking into consideration the formula

$$q = (1-\tau_1)^\beta\sqrt{2\alpha\tau_1}\,l_1$$

we obtain

$$\frac{l_1 - b}{l_1} = 4\tau_1\lambda\sin\lambda\sum_{n=1}^{\infty}\frac{(-)^n}{\pi^2 n^2 - \lambda^2}\left[\frac{z_{n1}'(\tau_1)}{z_{n1}(\tau_2)} + \frac{z_{n2}'(\tau_1)}{z_{n2}(\tau_1)}\right] \qquad (1.10)$$

Let us now calculate the difference between the distances of the points B and C from the axis of the stream; denoting the distance of the point C from the axis of the stream by c, we get

$$b - c = \int_{\tau_1}^{\tau_2}\frac{\partial y}{\partial \varphi}\frac{\partial \varphi}{\partial \tau}\,d\tau$$

Making use of the formulas

$$\int_{\tau_1}^{\tau_2}\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{d\tau}{\sqrt{2\alpha\tau}} = -\left[\frac{1}{(1-\tau)^\beta\sqrt{2\alpha\tau}}\right]_{\tau_1}^{\tau_2}$$

$$\left(1-\frac{\lambda^2}{\pi^2 n^2}\right)\int_{\tau_1}^{\tau_2}\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{dz_n}{d\tau}\right]\frac{d\tau}{\sqrt{2\alpha\tau}} = \left[\frac{\tau z_n'(\tau)}{(1-\tau)^\beta\sqrt{2\alpha\tau}} + \frac{1}{2}\frac{z_n(\tau)}{(1-\tau)^\beta\sqrt{2\alpha\tau}}\right]_{\tau_1}^{\tau_2}$$

and expanding (1.9), we obtain the following expression for $b - c$:

$$b - c = -q\left[\frac{1}{(1-\tau)^\beta\sqrt{2\alpha\tau}}\right]_{\tau_1}^{\tau_2} + 4\tau_2 l_2\lambda\sin\lambda\sum_{n=1}^{\infty}\frac{(-)^n}{\pi^2 n^2 - \lambda^2}\left[\frac{z_{n1}'(\tau_2)}{z_{n1}(\tau_2)} + \frac{1}{z_{n2}(\tau_1)}\right] -$$

$$-4\tau_1 l_1 \lambda \sin\lambda \sum_{n=1}^{\infty} \frac{(-)^n}{\pi^2 n^2 - \lambda^2}\left[\frac{1}{z_{n1}(\tau_2)} + \frac{z'_{n2}(\tau_1)}{z_{n2}(\tau_1)}\right] \tag{1.11}$$

Let us now develop a formula for the determination of the width of the jet going off to infinity. We have the following relation:

$$c - l_2 = \int_{\lambda}^{0} \frac{\partial y}{\partial \varphi}\frac{\partial \varphi}{\partial \theta}\, d\theta$$

Substituting therein for $\partial y/\partial \phi$ its value of $\sin\theta/\sqrt{2a\tau}$, and for $\partial \phi/\partial \theta$ its value, found from formula (1.9), we obtain

$$\frac{c - l_2}{l_2} = -4\tau_2 l_2 \lambda \sin\lambda \sum_{n=1}^{\infty} \frac{(-)^n}{\pi^2 n^2 - \lambda^2}\left[\frac{z'_{n1}(\tau_2)}{z_{n1}(\tau_2)} + \frac{z'_{n2}(\tau_2)}{z_{n2}(\tau_1)}\right] \tag{1.12}$$

We notice that if we combine the formulas (1.10), (1.11) and (1.12), then after cancelling similar terms we obtain an identity. Hence it follows, as indeed it should, that there is only one formula to determine l_2, namely,

$$l_2 = \left(\frac{1-\tau_1}{1-\tau_2}\right)^{\beta} \sqrt{\frac{\tau_1}{\tau_2}}\, l_1 \tag{1.13}$$

Since

$$\frac{d}{d\tau}\left\{(1-\tau)^{\beta}\sqrt{\tau}\right\} = \frac{(1-\tau)^{\beta-1}}{2\sqrt{\tau}}\left[1-(2\beta+1)\tau\right] > 0$$

and since for subsonic flows τ_1 and $\tau_2 < (2\beta+1)^{-1}$, then $l_2 < l_1$.

3. The formulas of the two foregoing subsections represent the complete formal solution of the specified problem on gaseous jets. The solution obtained, however, is in a very abstruse form, and accordingly we now turn to the deduction of simple relations which will enable us to determine the required dimensions, under the assumption that the velocities V_1 and V_2 are close to one another. Under this assumption the formulas (1.10), (1.11) and (1.12) can be reduced to a very simple form, by employing the transformations applied by Poincare [2] to the study of the propagation of radio waves.

Let us consider the following functions of the variable index n:

$$f_1(n) = \frac{1}{z_{n1}(\tau_2)}, \qquad f_2(n) = \frac{z'_{n2}(\tau_1)}{z_{n2}(\tau_1)}$$

$$f_3(n) = \frac{z'_{n1}(\tau_2)}{z_{n1}(\tau_2)}, \qquad f_4(n) = \frac{1}{z_{n2}(\tau_1)}$$

We notice that, by virtue of the relation

$$z_{n1}(\tau_2) = -z_{n2}(\tau_1)\left(\frac{1-\tau_2}{1-\tau_1}\right)^\beta \frac{\tau_1}{\tau_2}$$

we can express the function $f_4(n)$ by means of the function $f_1(n)$ according to the formula

$$f_4(n) = -\frac{\tau_1}{\tau_2}\left(\frac{1-\tau_2}{1-\tau_1}\right)^\beta f_1(n)$$

Since the parameter n^2 enters the equation (1.7) linearly, and the determination of the functions $z_{n1}(\tau)$ and $z_{n2}(\tau)$ obeys specially determined initial conditions, then according to one of Poincare's theorems [3] the functions $z_{n1}(\tau)$ and $z_{n2}(\tau)$ are entire functions of the variable n. Hence it follows that the function $f_1(n)$, $f_2(n)$, $f_3(n)$ and $f_4(n)$ are meromorphic functions of the complex variable n. Our first problem consists in expanding these functions in series as regards the principal parts.

Let us first consider the function $f_1(n)$ and find its poles. The affix n_j of a pole is a value of n for which the function $z_{n1}(\tau_2)$ vanishes; but according to its construction the function $z_{n1}(\tau)$ also vanishes when $\tau = \tau_1$. Consequently, n_j is a number such that, simultaneously,

$$z_{n_j 1}(\tau_1) = 0, \qquad z_{n_j 1}(\tau_2) = 0 \tag{1.14}$$

i.e. n_j, or rather $\pi^2 n_j^2/\lambda^2$, are the fundamental numbers of the differential equation (1.7) for the boundary conditions (1.14). Since the variable τ does not exceed $1/(2\beta+1)$, then the fundamental number $\pi^2 n_j^2/\lambda^2$ can only be a negative number. Let us introduce the real number m_j by setting $n_j = im_j$; then $\pi^2 n_j^2/\lambda^2 = -\pi^2 m_j^2/\lambda^2$, where the index j take the values $\pm 1, \pm 2, \pm 3, \ldots$ We observe that $m_{-j} = -m_j$.

For the further study of the function $f_1(n)$ it is convenient to transform equation (1.7) into a new form.

Let us introduce, instead of z, a new unknown function u, by setting

$$u = F(\tau) z$$

and in place of τ a new independent variable ν by the formula

$$\nu = \int_{\tau_1}^{\tau} \frac{\sqrt{1-(2\beta+1)\tau}}{\tau\sqrt{1-\tau}}\, d\tau \qquad F(\tau) = \left[\frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}}\right]^{1/4}$$

The function $u(\nu)$ will satisfy the equation

$$\frac{d^2u}{d\nu^2} - \left(\rho^2 + \frac{1}{F}\frac{d^2F}{d\nu^2}\right) u = 0 \tag{1.15}$$

where

$$\frac{1}{F}\frac{d^2F}{d\nu^2} = \frac{\beta(2\beta+1)}{4}\,\frac{\tau^2}{1-\tau}\,\frac{\beta(2\beta+1)\tau^2 + 2(\beta+2)\tau - 4}{[1-(2\beta+1)\tau]^3}, \qquad \rho^2 = \frac{\pi^2 n^2}{4\lambda^2}$$

Let us consider the solution of equation (1.15) for large $|\rho|$. According to a well-known theorem in the theory of differential operators [4], equation (1.15) has in each quadrant of the plane of the complex variable ρ two fundamental integrals $u^{(1)}(\nu)$ and $u^{(2)}(\nu)$, which, for large $|\rho|$ and for ν lying in the region of regularity of the function $(1/F)(d^2F/d\nu^2)$, are represented by the following asymptotic formulas:

$$u^{(1)}(\nu) = e^{\rho\nu}\left[1 + O\left(\frac{1}{\rho}\right)\right], \qquad u^{(2)}(\nu) = e^{-\rho\nu}\left[1 + O\left(\frac{1}{\rho}\right)\right]$$

Hence it follows that, as the parameter ρ varies in each of the specified quadrants, there exist fundamental integrals $z^{(1)}$ and $z^{(2)}$ for which the following asymptotic formulas hold:

$$z^{(1)}(\tau) = \frac{1}{F(\tau)} e^{\rho\nu}\left[1 + O\left(\frac{1}{\rho}\right)\right], \qquad z^{(2)}(\tau) = \frac{1}{F(\tau)} e^{-\rho\nu}\left[1 + O\left(\frac{1}{\rho}\right)\right]$$

This shows that in each of the specified quadrants there exist, for large $|\rho|$ the following asymptotic representations of the integral $z_{n1}(\tau)$ and its derivative:

$$z_{n1}(\tau) = \frac{e^{\rho\nu} - e^{-\rho\nu}}{\rho g_1 F(\tau_1)}, \qquad z_{n1}'(\tau) = \frac{e^{\rho\nu} + e^{-\rho\nu}}{g_1 F(\tau_1)} \frac{d\nu}{d\tau} \tag{1.16}$$

where

$$g_1 = \frac{2\sqrt{1 - (2\beta + 1)\tau_1}}{\tau_1 \sqrt{1 - \tau_1}\, F(\tau_1)} = \frac{2}{F(\tau_1)} \left(\frac{d\nu}{d\tau}\right)_{\tau_1}$$

Similarly, for the integral $z_{n2}(\tau)$ we have

$$z_{n2}(\tau) = \frac{e^{\rho(\nu - \nu')} - e^{-\rho(\nu - \nu')}}{\rho g_2 F(\tau_2)}, \quad z_{n2}'(\tau) = \frac{e^{\rho(\nu - \nu')} + e^{-\rho(\nu - \nu')}}{g_2 F(\tau_2)} \frac{d\nu}{d\tau} \tag{1.17}$$

where

$$g_2 = \frac{2\sqrt{1 - (2\beta + 1)\tau_2}}{\tau_2 \sqrt{1 - \tau_2}\, F(\tau_2)} = \frac{2}{F(\tau_2)} \left(\frac{d\nu}{d\tau}\right)_{\tau_2}, \qquad \nu' = \int_{\tau_1}^{\tau_2} \frac{\sqrt{1 - (2\beta + 1)\tau}}{\tau\sqrt{1 - \tau}}\, d\tau$$

Hence we obtain the following asymptotic representations of the functions $f_1(n)$, $f_2(n)$, $f_3(n)$, $f_4(n)$ for large $|\rho|$ or for large $|n|$ in all quadrants:

$$f_1(n) = \frac{2\sqrt{1 - (2\beta + 1)\tau_1}}{\tau_1\sqrt{1 - \tau_1}} \frac{\rho}{e^{\rho\nu'} - e^{-\rho\nu'}}, \qquad f_2(n) = \frac{\sqrt{1 - (2\beta + 1)\tau_1}}{\tau_1\sqrt{1 - \tau_1}} \rho \frac{e^{-\rho\nu'} + e^{\rho\nu'}}{e^{-\rho\nu'} - e^{\rho\nu'}}$$

$$f_3(n) = \frac{\sqrt{1 - (2\beta + 1)\tau_2}}{\tau_2\sqrt{1 - \tau_2}} \rho \frac{e^{\rho\nu'} + e^{-\rho\nu'}}{e^{\rho\nu'} - e^{-\rho\nu'}}, \qquad f_4(n) = \frac{2\sqrt{1 - (2\beta + 1)\tau_2}}{\tau_2\sqrt{1 - \tau_2}} \frac{\rho}{e^{-\rho\nu'} - e^{\rho\nu'}}$$

Starting from these formulas, we can establish the convergence of the series which appear in formulas (1.10), (1.11), and (1.12); using formulas (1.16) and (1.17), we can also demonstrate the convergence of the series (1.8) and (1.9), which define the stream function and the velocity potential.

Let us now carry out the expansion of the functions $f_1(n)$, $f_2(n)$, $f_3(n)$, $f_4(n)$ in series as regards the principal parts. We will start with the function $f_1(n)$.

The expansion of this function in series as regards the principal parts is

$$f_1(n) = \frac{1}{z_{01}(\tau_2)} + i \sum_{j=-\infty}^{\infty} \left(\frac{1}{n - im_j} + \frac{1}{im_j}\right)\xi_j$$

where the real number ξj is determined by the formula

$$\frac{1}{\xi_j} = i\left[\frac{\partial z_{n1}(\tau_2)}{\partial n}\right]_{n=im_j}$$

We observe that $\xi_{-j} = -\xi_j$.

Let us rewrite the foregoing expression for $f_1(n)$ in the form:

$$f_1(n) = \frac{1}{z_{01}(\tau_2)} + 2\sum_{j=1}^{\infty} \frac{\xi_j}{m_j} - 2\sum_{j=1}^{\infty} \frac{\xi_j m_j}{n^2 + m_j^2} \tag{1.18}$$

Using the relation between the functions $f_1(n)$ and $f_4(n)$, we can also write down the expansion as regards the principal parts of the function $f_4(n)$:

$$f_4(n) = \frac{z_0{}'_2(\tau_2)}{z_{02}(\tau_1)} + 2\sum_{j=1}^{\infty} \frac{\omega_j}{m_j} - 2\sum_{j=1}^{\infty} \frac{\omega_j m_j}{n^2 + m_j^2} \tag{1.19}$$

where

$$\omega_j = -\frac{\tau_1}{\tau_2}\left(\frac{1-\tau_2}{1-\tau_1}\right)^{\beta}\xi_j$$

Now let us expand the function $f_2(n)$ in series as regards the principal parts. Taking into consideration the asymptotic formula for $f_2(n)$, we find after some minor transformations that

$$f_2(n) = \frac{z_0{}'_2(\tau_1)}{z_{02}(\tau_1)} + 2\sum_{j=1}^{\infty} \frac{\eta_j}{m_j} - 2\sum_{j=1}^{\infty} \frac{\eta_j m_j}{n^2 + m_j^2} \tag{1.20}$$

where

$$i\eta_j = \left[\frac{z_{n2}'(\tau_1)}{(\partial/\partial n)\, z_{n2}(\tau_1)}\right]_{n=im_j}, \qquad \eta_{-j} = -\eta_j$$

Similarly for the function $f_3(n)$ we obtain this expansion:

$$f_3(n) = \frac{z_{01}'(\tau_2)}{z_{01}(\tau_2)} + 2\sum_{j=1}^{\infty} \frac{\zeta_j}{m_j} - 2\sum_{j=1}^{\infty} \frac{\zeta_j m_j}{n^2 + m_j^2} \tag{1.21}$$

where

$$i\zeta_j = \left[\frac{z'_{n1}(\tau_2)}{(\partial/\partial n)\, z_{n1}(\tau_2)}\right]_{n=im_j}$$

4. Let us now take the formulas (1.10) and (1.12) and transform them into a new form, replacing therein the functions $f_1(n)$, $f_2(n)$, $f_3(n)$, $f_4(n)$ by their expansions (1.18), (1.19), (1.20), and (1.21). We obtain the following results:

$$\sum_{n=1}^{\infty} \frac{(-)^n}{\pi^2 n^2 - \lambda^2} f_1(n) = \frac{1}{2\lambda}\left(\frac{1}{\lambda} - \operatorname{cosec}\lambda\right)\left\{\frac{z'_{01}(\tau_1)}{z_{01}(\tau_2)} + 2\lambda^2 \sum_{j=1}^{\infty} \frac{\xi_j}{m_j} \frac{1}{\lambda^2 + \pi^2 m_j^2}\right\} - $$
$$- \sum_{j=1}^{\infty} \frac{\xi_j}{m_j} \frac{1}{\lambda^2 + \pi^2 m_j^2}\left(1 - \frac{\pi m_j}{\operatorname{sh} \pi m_j}\right)$$

Now replacing ξ_j in the right-hand side successively by η_j, ζ_j, and ω_j, we shall have the new expressions respectively for

$$\sum_{n=1}^{\infty} \frac{(-)^n}{\pi^2 n^2 - \lambda^2} f_2(n), \quad \sum_{n=1}^{\infty} \frac{(-)^n}{\pi^2 n^2 - \lambda^2} f_3(n), \quad \sum_{n=1}^{\infty} \frac{(-)^n}{\pi^2 n^2 - \lambda^2} f_4(n)$$

Let us return now to formulas (1.10) and (1.12) and substitute in them the expressions obtained for the sums under consideration. Bearing in mind the equations

$$\frac{z'_{02}(\tau_1)}{z_{01}(\tau_2)} + \frac{z'_{02}(\tau_1)}{z_{02}(\tau_1)} = 0, \qquad \frac{z'_{01}(\tau_2)}{z_{01}(\tau_2)} + \frac{z'_{02}(\tau_2)}{z_{02}(\tau_2)} = 0$$

we obtain

$$\frac{c - l_2}{l_2} = 4\tau_2\lambda \sin\lambda(\pi S_1 - \lambda \csc\lambda\, S_2) \tag{1.22}$$

$$\frac{l_1 - b}{l_1} = 4\tau_1\lambda \sin\lambda\,(\pi S_3 - \lambda \csc\lambda S_4) \tag{1.23}$$

$$S_1 = \sum_{j=1}^{\infty} \frac{\zeta_j + \omega_j}{(\lambda^2 + \pi^2 m_j^2)\operatorname{sh}\pi m_j}, \qquad S_2 = \sum_{j=1}^{\infty} \frac{\zeta_j + \omega_j}{m_j} \frac{1}{\lambda^2 + \pi^2 m_j^2} \tag{1.24}$$

$$S_3 = \sum_{j=1}^{\infty} \frac{\xi_j + \eta_j}{(\lambda^2 + \pi^2 m_j^2)\operatorname{sh}\pi m_j}, \qquad S_4 = \sum_{j=1}^{\infty} \frac{\xi_j + \eta_j}{m_j} \frac{1}{\lambda^2 + \pi^2 m_j^2} \tag{1.25}$$

5. Formulas (1.22) and (1.23) can be reduced to an exceptionally simple form in the case when the numbers r_1 and r_2 are close to one another. In order to obtain these new formulas let us take equation (1.15) and rewrite it in the following way, replacing n by im :

$$\frac{d^2u}{d\nu^2}+\left(\frac{\pi^2m^2}{4\lambda^2}-\frac{1}{F}\frac{d^2F}{d\nu^2}\right)u=0$$

Let us introduce in place of ν the new independent variable

$$\xi=\frac{\pi}{\nu'}\nu$$

We observe that when the difference between r_1 and r_2 is not large the parameter ν' is very small. The last differential equation is rewritten in terms of the variable ξ as follows:

$$\frac{d^2u}{d\xi^2}+\left[s^2-\frac{\nu'^2}{\pi^2}\frac{1}{F}\frac{d^2F}{d\nu^2}\right]u=0 \qquad \left(s=\frac{m\nu'}{2\lambda}\right) \tag{1.26}$$

Let us calculate the integral of this equation under the following conditions:

$$u(0)=0, \qquad \left(\frac{du}{d\xi}\right)_{\xi=0}=1$$

The original variable r can be expressed in terms of the variable ξ in the form of the following series in powers of $\nu'\xi/\pi$:

$$\tau=\tau_1+\frac{\tau_1\sqrt{1-\tau_1}}{\sqrt{1-(2\beta+1)\tau_1}}\frac{\nu'\xi}{\pi}+\cdots$$

By virtue of this we obtain

$$\frac{1}{F}\frac{d^2F}{d\nu^2}=a_0+a_1\nu'\xi+a_2\nu'^2\xi^2+\ldots \qquad \left(a_0=\left[\frac{1}{F}\frac{d^2F}{d\nu^2}\right]_{\nu=0}\right).$$

Hence we can rewrite equation (1.26) in the following way:

$$\frac{d^2u}{d\xi^2}+s^2u=\frac{\nu'^2}{\pi^2}\left[a_0+a_1\nu'\xi+a_2\nu'^2\xi^2+\cdots\right]u(\xi) \tag{1.27}$$

We shall seek a solution of this equation in the form of a series in powers of the parameter ν'; we set

$$u(\xi)=u_0(\xi)+\nu'u_1(\xi)+\nu'^2u_2(\xi)+\nu'^3u_3(\xi)+\ldots \tag{1.28}$$

In order to determine the co-efficients $u_0(\xi)$, $u_1(\xi)$, ... we have the following system of equations:

$$\frac{d^2u_0}{d\xi^2}+s^2u_0=0, \quad \frac{d^2u_1}{d\xi^2}+s^2u_1=0, \quad \frac{d^2u_2}{d\xi^2}+s^2u_2=\frac{a_0}{\pi^2}u_0$$

$$\frac{d^2u_3}{d\xi^2}+s^2u_3=\frac{1}{\pi^2}(a_1\xi u_0+a_0u_1), \qquad \frac{d^2u_4}{d\xi^2}+s^2u_4=\frac{1}{\pi^2}(a_0u_2+a_1\xi u_1+a_2\xi^2u_0),\ldots$$

This system of equations has to be integrated under the following conditions:

$$u_0(0)=0, \quad u_1(0)=0, \quad u_2(0)=0, \quad u_3(0)=0, \quad u_4(0)=0,\ldots$$
$$u_0'(0)=1, \quad u_1'(0)=1, \quad u_2'(0)=0, \quad u_3'(0)=0, \quad u_4'(0)=0,\ldots$$

Integrating the system of equations so obtained under these conditions, we have

$$u_0(\xi) = \frac{1}{s}\sin s\xi, \quad u_1(\xi) = 0, \quad u_2(\xi) = -\frac{a_0}{2\pi^2 s^3}[s\xi\cos s\xi - \sin s\xi]$$

$$u_3(\xi) = -\frac{a_1\xi}{4\pi^2 s^3}[s\xi\cos s\xi - \sin s\xi]$$

$$u_4(\xi) = \frac{1}{8\pi^4 s^3}(2\pi^2 a_2 - a_0^2)\,\xi^2\sin s\xi - \frac{a_2}{6\pi^2 s^2}\xi^3\cos s\xi +$$
$$+\frac{1}{8\pi^4 s^4}(2\pi^2 a_2 - 3a_0^2)\,\xi\cos s\xi - \frac{1}{8\pi^4 s^4}(2\pi^2 a_2 - 3a_0^2)\frac{\sin s\xi}{s}$$

In this way the series (1.28) is constructed. Let us find the number s from the condition that the solution (1.28) of equation (1.26) shall vanish when $\xi = \pi$.

The equation for the determination of s is written thus: (1.29)

$$\sin\pi s - \frac{a_0}{2\pi^2 s^2}[\pi s\cos\pi s - \sin\pi s]\nu'^2 - \frac{a_1\pi}{4\pi^2 s^2}[\pi s\cos\pi s - \sin\pi s]\,\nu'^3 + \ldots = 0$$

When $\nu' = 0$ this equation has the solution $s = j$, where j is an arbitrary integer. The partial derivative with respect to s of the left-hand side of this equation differs from zero when $\nu' = 0$ and $s = j$; consequently, equation (1.29) has a holomorphic solution when $\nu' \neq 0$; this solution can be represented up to and including second degree terms in ν' by the series:

$$s = j + \frac{a_0}{2\pi^2 j}\nu'^2 + \ldots \tag{1.30}$$

On the basis of the foregoing calculations, we can write down $u(\xi)$ in explicit form thus:

$$u_{n1}(\xi) = \frac{1}{s}\sin s\xi - \frac{a_0}{2\pi^2 s^3}(s\xi\cos s\xi - \sin s\xi)\,\nu'^2 + \ldots \tag{1.31}$$

Let us take equation (1.26) once again and find the integral of it which satisfies the conditions

$$u(\pi) = 0, \qquad \frac{du}{d\xi}(\pi) = 1$$

Let us set

$$u(\xi) = u_0(\xi) + \nu' u_1(\xi) + \nu'^2 u_2(\xi) + \ldots \tag{1.32}$$

In order to determine the new functions u_0, u_1, u_2, ... we shall have the previous equations, but the boundary conditions are different, namely:

$$u_0(\pi) = 0, \qquad u_1(\pi) = 0, \qquad u_2(\pi) = 0, \ldots$$
$$u_0'(\pi) = 1, \qquad u_1'(\pi) = 0, \qquad u_2'(\pi) = 0, \ldots$$

Under these conditions the new solution of equation (1.26) can be constructed in the form of the following series:

$$u_{n2}(\xi)=\frac{1}{s}\sin s(\xi-\pi)-\frac{a_0}{2\pi^2s^3}[s(\xi-\pi)\cos s(\xi-\pi)-\sin s(\xi-\pi)]\nu'^2+\ldots \quad (1.33)$$

We need to have the functions z_{n1} and z_{n2}. These functions are connected with the functions u_{n1} and u_{n2}, which are given respectively by formulas (1.31) and (1.33) and satisfy the boundary conditions, by the relations

$$z_{n1}(\tau)=h(\tau_1)\frac{u_{n1}}{F(\tau)}\frac{\nu'}{\pi},\quad z_{n2}(\tau)=h(\tau_2)\frac{u_{n2}}{F(\pi)}\frac{\nu'}{\pi},\quad h(\tau)=F(\tau)\frac{d\tau}{d\nu} \quad (1.34)$$

Hence, using formulas (1.30), (1.31) and the relation

$$s=-i\frac{n\nu'}{2\lambda}$$

we obtain the following result:

$$\frac{\partial z_{n1}(\xi)}{\partial n}=\frac{\nu'}{\pi}\frac{h(\tau_1)}{F(\tau_2)}\Big\{\frac{s\xi\cos s\xi-\sin s\xi}{s^2}-$$
$$-\frac{a_0}{2\pi^2s^4}[(3-s^2\xi^2)\sin s\xi-3s\xi\cos s\xi]\nu'^2+\ldots\Big\}\Big(-\frac{i\nu'}{2\lambda}\Big)$$

Now, setting $\xi=\pi$ and replacing s by the expression (1.30), we obtain

$$\left[\frac{\partial z_{n1}(\tau_2)}{\partial n}\right]_{n=im_j}=(-)^{j+1}\frac{i\nu'^2}{2\lambda\pi}\frac{h(\tau_1)}{F(\tau_2)}\frac{\pi}{j}\Big\{1+\frac{3a_0}{2\pi^2}\frac{\nu'^2}{j^2}+\ldots\Big\} \quad (1.35)$$

Making use of the second formula (1.33), we find by similar manipulations that

$$\left[\frac{\partial z_{n2}(\tau_1)}{\partial n}\right]_{n=im_j}=(-)^{j}\frac{i\nu'^2}{2\lambda\pi}\frac{h(\tau_2)}{F(\tau_1)}\frac{\pi}{j}\Big\{1+\frac{3a_0}{2\pi^2}\frac{\nu'^2}{j^2}+\ldots\Big\} \quad (1.36)$$

The formulas (1.35) and (1.36) so obtained make it possible to find the numbers ξ_j, η_j, ζ_j, ω_j, introduced in subsection 3. We have

$$\xi_j=(-)^j\frac{2\lambda j}{\nu'^2}\frac{F(\tau_2)}{h(\tau_1)}\Big\{1-\frac{3a_0}{2\pi^2}\frac{\nu'^2}{j^2}+\ldots\Big\}$$
$$\eta_j=-\frac{2\lambda j}{\nu'^2}\frac{F(\tau_1)}{h(\tau_1)}\Big\{1-\frac{3a_0}{2\pi^2}\frac{\nu'^2}{j^2}+\ldots\Big\}$$
$$\zeta_j=\frac{2\lambda j}{\nu'^2}\frac{F(\tau_2)}{h(\tau_2)}\Big\{1-\frac{3a_0}{2\pi^2}\frac{\nu'^2}{j^2}+\ldots\Big\}$$
$$\omega_j=-\frac{\tau_1}{\tau_2}\Big(\frac{1-\tau_2}{1-\tau_1}\Big)^\beta\xi_j=(-)^{j+1}\frac{2\lambda j}{\nu'^2}\frac{F(\tau_1)}{h(\tau_2)}\Big\{1-\frac{3a_0}{2\pi^2}\frac{\nu'^2}{j^2}+\ldots\Big\}$$

6. Let us now use these expressions to evaluate formulas (1.22) and (1.23). First of all we find the sum of the series S_4 for small ν'. We have

$$\sum_{j=1}^{\infty}\frac{\xi_j}{m_j}\frac{1}{\lambda^2+\pi^2m_j^2}=-\frac{\nu'}{48\lambda^2}\frac{F(\tau_2)}{h(\tau_1)}\Big\{1-\frac{7(12a_0+1)}{240}\nu'^2+\ldots\Big\}$$
$$\sum_{j=1}^{\infty}\frac{\eta_j}{m_j}\frac{1}{\lambda^2+\pi^2m_j^2}=-\frac{\nu'}{24\lambda^2}\frac{F(\tau_1)}{h(\tau_1)}\Big\{1-\frac{12a_0+1}{60}\nu'^2+\ldots\Big\}$$

Hence

$$S_4 = -\frac{\nu'}{24\lambda^2}\left\{\frac{F(\tau_1)}{h(\tau_1)} + \frac{1}{2}\frac{F(\tau_2)}{h(\tau_1)}\right\} + \frac{12a_0+1}{24\cdot 60}\frac{\nu'^3}{\lambda^2}\left\{\frac{F(\tau_1)}{h(\tau_1)} + \frac{7}{8}\frac{F(\tau_2)}{h(\tau_1)}\right\} + \ldots \quad \text{(a)}$$

The expression for the sum of the series S_2, as determined by formula (1.24) for small ν', has the following form:

$$S_2 = \frac{\nu'}{24\lambda^2}\left\{\frac{F(\tau_2)}{h(\tau_2)} + \frac{1}{2}\frac{F(\tau_1)}{h(\tau_2)}\right\} - \frac{12a_0+1}{24\cdot 60}\frac{\nu'^3}{\lambda^2}\left\{\frac{F(\tau_2)}{h(\tau_2)} + \frac{7}{8}\frac{F(\tau_1)}{h(\tau_2)}\right\} + \ldots \quad \text{(б)}$$

If the number ν' is small, as we are assuming, then the infinite sums S_1 and S_3, determined by formulas (1.24) and (1.25), are significantly smaller than the sums S_2 and S_4.

Therefore, substituting in formulas (1.22) and (1.23) the expressions (a) and (b) in place of S_2 and S_4, and neglecting the terms containing the sums S_1 and S_3, we obtain

$$\frac{c-l_2}{l_2} = \frac{\tau_2\nu'}{6}\left\{\frac{F(\tau_2)}{h(\tau_2)} + \frac{1}{2}\frac{F(\tau_1)}{h(\tau_2)}\right\} - \frac{12a_0+1}{360}\tau_2\nu'^3\left\{\frac{F(\tau_2)}{h(\tau_2)} + \frac{7}{8}\frac{F(\tau_1)}{h(\tau_2)}\right\} + \ldots$$

$$\frac{l_1-b}{l_1} = \frac{\tau_1\nu'}{6}\left\{\frac{F(\tau_1)}{h(\tau_1)} + \frac{1}{2}\frac{F(\tau_2)}{h(\tau_1)}\right\} - \frac{12a_0+1}{360}\tau_1\nu'^3\left\{\frac{F(\tau_1)}{h(\tau_1)} + \frac{7}{8}\frac{F(\tau_2)}{h(\tau_1)}\right\} + \ldots$$

Bearing in mind the value of the function $h(\tau)$, we can transform these two formulas into the following form:

$$\frac{c-l_2}{l_2} = \frac{\nu'}{6}\left\{\left[1 + \frac{1}{2}\frac{F(\tau_1)}{F(\tau_2)}\right] - \frac{1+12a_0}{60}\nu'^2\left[1 + \frac{7}{8}\frac{F(\tau_1)}{F(\tau_2)}\right] + \ldots\right\}\frac{\sqrt{1-(2\beta+1)\tau_2}}{\sqrt{1-\tau_2}} \quad (1.37)$$

$$\frac{l_1-b}{l_1} = \frac{\nu'}{6}\left\{\left[1 + \frac{1}{2}\frac{F(\tau_2)}{F(\tau_1)}\right] - \frac{1+12a_0}{60}\nu'^2\left[1 + \frac{7}{8}\frac{F(\tau_2)}{F(\tau_1)}\right] + \ldots\right\}\frac{\sqrt{1-(2\beta+1)\tau_1}}{\sqrt{1-\tau_1}}$$

Let us take as our fundamental data in this problem the quantities τ_1, l_1, c; then from formulas (1.13) and (1.37) we can determine τ_2, l_2 and b. If we know the value of the speed of sound in the inflowing gas, then by the same token we have the value of the parameter a, and can therefore determine the velocity of the gas issuing from the orifice.

If in formulas (1.37) we set $\beta = 0$, then we obtain the formulas relating to incompressible fluid.

In this case we have $F(\tau) = 1$, $a_0 = 0$ and the foregoing formulas assume the following form

$$\frac{c-l_2}{l_2} = \frac{1}{6}\nu'\left(\frac{3}{2} - \frac{1}{32}\nu'^2 + \ldots\right)$$

$$\frac{l_1-b}{l_1} = \frac{1}{6}\nu'\left(\frac{3}{2} - \frac{1}{32}\nu'^2 + \ldots\right)$$

These two equations express Zhukovskii's theorem [6]:

$$\frac{l_1 - b}{l_1} = \frac{c - l_2}{l_2}$$

It must be pointed out that this theorem is valid for incompressible fluid for arbitrary τ_1 and τ_2, and in this general form it can be obtained from the complete formulas (1.10) and (1.12).

II. The pressure of a gaseous stream on a flat plate

7. The method described in the previous section can be used to solve the problem of a jet flowing past a flat plate under the assumption that the point of zero velocity, occurring at the plate, is replaced by a wedge-shaped region of stagnant fluid [7].

Accordingly, let us assume that a stream of gas, having a velocity at infinity of V_2 and a width of $2L$, impinges on a plate of length $2l$, disposed symmetrically relative to the gas stream. Let us further assume that the stream separates from the ends of the plate with free streamlines, and that it also forms, at the middle of the plate, a stagnant region along the curvilinear boundaries of which the particle velocity of the gas is constant and equal to V_1. This stagnant region replaces the point of zero velocity which usually occurs at the centre of the plate. Our problem consists in determining the pressure of the stream on the plate and in finding the various geometrical quantities connected with the flow pattern under consideration.

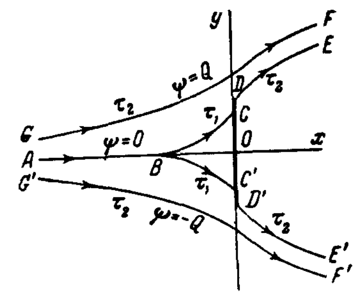

Fig. 4.

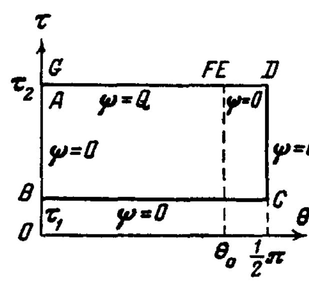

Fig. 5.

In Fig. 4 the streamline GF and $G'F'$ are symmetrically disposed relative to the axis Ox and enclose the whole moving mass of gas; the streamlines DE and $D'E'$ spring from the ends D and D' of the plate; the parts BC and BC' of the complete streamlines $ABCDE$ and $ABC'D'E'$ replace the point of zero velocity. On the streamlines DE and $D'E'$ the velocity of flow is constant and equal to V_2; on the streamlines BC, BC', the velocity is likewise constant and equal to $V_1 < V_2$.

Let the streamline $ABCDE$ correspond to the zero value of the stream function $\psi(\tau, \theta)$, and the streamline GF correspond to the value of the stream function equal to $Q > 0$.

Let us denote by θ_0 the angle which is formed by the direction of the divided jet with the axis of x; this quantity is unknown. In Fig. 5, representing the plane of Chaplygin's variables, the flow occupies the region $ABCDEFGA$, inside which we require the integral of the equation

$$\frac{\partial}{\partial\tau}\left\{\frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}\right\} + \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{2.1}$$

with the given values of ψ on the contour of this region.

We will seek the function $\psi(\tau, \theta)$ in the form of the following infinite series:

$$\psi(\tau, \theta) = \sum_{n=1}^{\infty} A_n \frac{z_n(\tau)}{z_n(\tau_2)} \sin 2n\theta \tag{2.2}$$

where the co-efficients A_n are to be determined, and the function $z_n(\tau)$ is the integral of the equation

$$\frac{d}{d\tau}\left\{\frac{\tau}{(1-\tau)^{\beta}}\frac{dz}{d\tau}\right\} - n^2\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}} z = 0 \tag{2.3}$$

and satisfies the following requirements:

$$z_n(\tau_1) = 0, \qquad \left(\frac{dz_n}{d\tau}\right)_{\tau_1} = 1$$

The coefficients A_n have to be found from the conditions:

$$\sum_{n=1}^{\infty} A_n \sin 2n\theta = \begin{cases} Q & (0 < \theta < \theta_0) \\ 0 & (\theta_0 < 0 < \frac{1}{2}\pi) \end{cases}$$

We find that

$$A_n = \frac{4Q}{\pi n}\sin^2 n\theta_0$$

Accordingly, for the stream function $\psi(\tau, \theta)$ we obtain the following expansion:

$$\psi(\tau, \theta) = \frac{4Q}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2 n\theta_0}{n}\frac{z_n(\tau)}{z_n(\tau_2)}\sin 2n\theta \tag{2.4}$$

Hence we derive the following expression for the velocity potential:

$$\varphi(\tau, \theta) = C - \frac{4Q\tau}{\pi(1-\tau)^{\beta}}\sum_{n=1}^{\infty}\frac{\sin^2 n\theta_0}{n^2}\frac{z_n'(\tau)}{z_n(\tau_2)}\cos 2n\theta \tag{2.5}$$

8. Let us now calculate the lengths OC and CD of the plate, using the formulas obtained in the previous subsection.

To calculate the length OC, we observe that along the streamline BC the following formula is valid:

$$dy = \frac{\sin\theta}{\sqrt{2\alpha\tau}} \frac{\partial\varphi}{\partial\theta} d\theta$$

Integrating both sides of this formula with respect to θ from 0 to $1/2\,\pi$, after replacing τ by τ_1, we find with the help of formula (2.5) that

$$OC = \frac{16Q}{\pi} \frac{\tau_1}{\sqrt{2\alpha\tau_1}\,(1-\tau_1)^\beta} \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{4n^2-1} \frac{1}{z_n(\tau_2)} \sin^2 n\theta_0 \tag{2.6}$$

Along the segment CD of the plate, we know that

$$dy = \frac{\sin\theta}{\sqrt{2\alpha\tau}} \frac{\partial\varphi}{\partial\tau} d\tau, \qquad \theta = \tfrac{1}{2}\pi$$

Integrating both sides of this equation with respect to the variable τ from τ_1 up to τ_2, we obtain:

$$CD = \frac{4Q}{\pi\sqrt{2\alpha}} \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n^2} \frac{\sin^2 n\theta_0}{z_n(\tau_2)} \int_{\tau_1}^{\tau_2} \frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{dz_n}{d\tau}\right]\frac{d\tau}{\sqrt{\tau}}$$

or, effecting the integration,

$$CD = \frac{16Q}{\pi\sqrt{2\alpha}} \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{4n^2-1} \frac{\sin^2 n\theta_0}{z_n(\tau_2)} \left\{\frac{1}{(1-\tau_2)^3}\frac{d}{d\tau_2}\left[\sqrt{\tau_2}z_n(\tau_2)\right] - \frac{\sqrt{\tau_1}}{(1-\tau_1)^\beta}\right\} \tag{2.7}$$

Combining the formula so obtained with formula (2.6) and making use of the relation

$$\sum_{n=1}^{\infty} \frac{(-)^{n-1}}{4n^2-1} \sin^2 n\theta_0 = \tfrac{1}{4}\pi \sin^2 \tfrac{1}{2}\theta_0 \tag{2.8}$$

we find that

$$2l = \frac{32Q}{\pi\sqrt{2\alpha\tau_2}\,(1-\tau_2)^\beta} \left\{\tfrac{1}{8}\pi\sin^2\tfrac{1}{2}\theta_0 + \tau_2 \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{4n^2-1} \frac{z'_n(\tau_2)}{z_n(\tau_2)} \sin^2 n\theta_0 \right. \tag{2.9}$$

9. Let us calculate the magnitude of the pressure of the stream on the plate. We shall denote by p_2 the pressure at infinity in the region of stationary gas behind the plate, and by p_1 the pressure of the gas in the stagnant region in front of the plate. Then the resultant R of the pressure forces of the gas on the plate is

$$R = 2\int_{CD} p\,dy - 2p_2\,CD + 2(p_1 - p_2)\,OC \tag{2.10}$$

or

$$R = 2\int_{CD} p\,dy - 2p_2 l + 2p_1\, OC \qquad \left(p = p_1\left[\frac{1-\tau}{1-\tau_1}\right]^{\beta+1}\right)$$

Along the segment CD of the wetted area of the plate we have

$$dy = \frac{4Q}{\pi}\frac{1}{\sqrt{2\alpha\tau}}\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n^2}\frac{\sin^2 n\theta_0}{z_n(\tau_2)}\frac{d}{d\tau}\frac{\tau z_n'(\tau)}{(1-\tau)^{\beta}}\,d\tau$$

Hence we obtain

$$\int_{CD} p\,dy = \frac{4Q}{\pi\sqrt{2\alpha}}\frac{p_1}{(1-\tau_1)^{\beta+1}}\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n^2}\frac{\sin^2 n\theta_0}{z_n(\tau_2)}\int_{\tau_1}^{\tau_2}\frac{(1-\tau)^{\beta+1}}{\sqrt{\tau}}\frac{d}{d\tau}\frac{\tau z_n'(\tau)}{(1-\tau)^{\beta}}\,d\tau$$

But

$$\int_{\tau_1}^{\tau_2}\frac{(1-\tau)^{\beta+1}}{\sqrt{\tau}}\frac{d}{d\tau}\frac{\tau z_n'(\tau)}{(1-\tau)^{\beta}}\,d\tau = \frac{4n^2}{4n^2-1}(1-\tau_2)\frac{d}{d\tau_2}\left[\sqrt{\tau_2}\, z_n(\tau_2)\right] +$$

$$+\frac{4n^2}{4n^2-1}(\beta+1)\sqrt{\tau_2}\, z_n(\tau_2) - \frac{4n^2}{4n^2-1}\sqrt{\tau_1}\,(1-\tau_1)$$

and therefore

$$\int_{CD} p\,dy = p_1 l\left(\frac{1-\tau_2}{1-\tau_1}\right)^{\beta+1} - p_1\, OC + \frac{4Q\, p_1}{(1-\tau_1)^{\beta+1}}\sqrt{\frac{\tau_2}{2\alpha}}\sin^2\tfrac{1}{2}\theta_0$$

Let us now evaluate the expression for R in formula (2.10); we eventually obtain the following result for R:

$$R = \frac{8Q(\beta+1)}{(1-\tau_1)^{\beta+1}}\sqrt{\frac{\tau_2}{2\alpha}}\sin^2\tfrac{1}{2}\theta_0\, p_1 \tag{2.11}$$

Accordingly, the problem which we formulated has been solved. Given the quantities τ_1, τ_2, l and Q, we can calculate the angle θ_0 from formula (2.9), and the length of the wetted portion of the plate from formula (2.7); then the magnitude of the pressure of the stream upon the plate is given by formula (2.11).

10. From the formulas of the foregoing subsection let us recover Chaplygin's results for incompressible fluid in the flow pattern under consideration.

In this case we have

$$\beta = 0, \qquad z_n(\tau) = \frac{\tau_1}{2n}\left(\frac{\tau^n}{\tau_1^{\,n}} - \frac{\tau_1^{\,n}}{\tau^n}\right)$$

$$z_n'(\tau) = \frac{\tau_1}{2}\left(\frac{\tau^{n-1}}{\tau_1^{\,n}} - \frac{\tau_1^{\,n}}{\tau^{n+1}}\right)$$

Let us evaluate formulas (2.6) and (2.9): we obtain

$$OC = \frac{32Q}{\pi V_1} \sum_{n=1}^{\infty} \frac{(-)^{n-1}n}{4n^2-1} \frac{q^n \sin^2 n\theta_0}{1-q^{2n}} \tag{2.12}$$

$$2l = \frac{32Q}{\pi V_2}\left(\frac{1}{8}\pi \sin^2 \frac{1}{2}\theta_0 + \sum_{n=1}^{\infty} \frac{(-)^{n-1}n}{4n^2-1} \frac{1+q^{2n}}{1-q^{2n}} \sin^2 n\theta_0\right) \tag{2.13}$$

where

$$q = \frac{\tau_1}{\tau_2} = \left(\frac{V_1}{V_2}\right)^2 < 1$$

For incompressible fluid we have

$$\alpha = \frac{p_0}{\rho}, \qquad \frac{p_1}{\rho} = \frac{p_0}{\rho} - \frac{1}{2}V_1^2, \qquad \tau = \frac{\rho V^2}{2p_0}$$

where p_0 is the stagnation pressure, p_1 is the pressure at velocity V_1. By virtue of these formulas we find that for incompressible fluid

$$\frac{\beta+1}{(1-\tau_1)^{\beta+1}} \sqrt{\frac{\tau_2}{2\alpha}}\, p_1 = \left(1 - \frac{\rho V_1^2}{2p_0}\right)^{-1} \frac{\rho V_2}{2p_0}\, p_1 = \frac{1}{2}\rho V_2$$

Hence, formula (2.11) assumes the form:

$$R = 4Q\rho V_2 \sin^2 \frac{1}{2}\theta_0 \tag{2.14}$$

The collection of formulas (2.12), (2.13) and (2.14) in fact solves the stipulated problem of jet flow of an incompressible liquid. From these formulas we can eliminate θ_0 and obtain the Chaplygin formulas in the case when the incident stream has infinite width; in that case $Q = \infty$. It is evident from formula (2.13) that, when Q tends to infinity, the angle θ_0 approaches indefinitely close to zero. Let us investigate the law governing this approach to the zero limit. For this purpose let us find the sum of the infinite series in formula (2.13) for small θ_0. We have

$$\sum_{n=1}^{\infty} \frac{(-)^{n-1}n}{4n^2-1} \frac{1+q^{2n}}{1-q^{2n}} \sin^2 n\theta_0 = -\frac{1}{8}\ln\cos\theta_0 + \frac{\theta_0^2}{4} \sum_{n=1}^{\infty} \frac{(-)^{n-1}n}{4n^2-1} \frac{\sin^2 n\theta_0}{n^2\theta_0^2} + $$
$$+ 2\theta_0^2 \sum_{n=1}^{\infty} \frac{(-)^{n-1}n^3}{4n^2-1} \frac{q^{2n}}{1-q^{2n}} \frac{\sin^2 n\theta_0}{n^2\theta_0^2}$$

Hence it follows that for small θ_0,

$$\sum_{n=1}^{\infty} \frac{(-)^{n-1}n}{4n^2-1} \frac{1+q^{2n}}{1-q^{2n}} \sin^2 n\theta_0 = \left\{\frac{1}{8} + 2\sum_{n=1}^{\infty} \frac{(-)^{n-1}n^3}{4n^2-1} \frac{q^{2n}}{1-q^{2n}}\right\}\theta_0^2$$

Now, for small θ_0 formula (2.13) gives the following result:

$$Q\theta_0^2 = \frac{2\pi l V_2}{\pi + 4 + 64S}, \qquad S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{4n^2-1} \frac{q^{2n}}{1-q^{2n}}$$

Substituting this expression for $Q\theta_0^2$ in formulas (2.12) and (2.14), we obtain the Chaplygin results [7]:

$$OC = \frac{64l}{\sqrt{q}} \frac{T}{\pi + 4 + 64S}, \qquad R = \frac{2\pi\rho l V_2^2}{\pi + 4 + 64S} \qquad T = \sum_{n=1}^{\infty} \frac{(-)^{n-1}n^3}{4n^2 - 1} \frac{q^n}{1-q^{2n}}$$

Let us consider one more particular case. Let us assume that the region of zero velocity is absent from the front of the plate; in this case the number q is equal to zero and formula (2.13) takes the following form:

$$2l = \frac{32Q}{\pi V_2}\left(\frac{1}{8}\pi \sin^2 \frac{1}{2}\theta_0 + \sum_{n=1}^{\infty} \frac{(-)^{n-1} n}{4n^2 - 1} \sin^2 n\theta_0\right)$$

or, after summing the infinite series,

$$l = \frac{2Q}{\pi V_2}\left[\pi \sin^2 \frac{1}{2}\theta_0 + \sin\theta_0 \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{\theta_0}{2}\right)\right]$$

Using this equation to determine θ_0 from l, Q and V_2, we obtain from formula (2.14) the pressure of the stream on the plate:

$$R = \frac{2\pi\rho l V_2^2}{\pi + \operatorname{ctg} {}^1/_2\theta_0 \ln[(1 + \sin\theta_0)/(1 - \sin\theta_0)]}$$

This expression for R agrees with that obtained by Zhukovskii [6], section 10.

11. Let us now return to the general formulas of subsections 8 and 9 and find the geometrical dimensions and the force R for the case when the velocities V_1 and V_2 are close to one another.

For the analysis of formulas (2.6) and (2.9) we need to consider the dependence upon n of the two functions:

$$\frac{1}{z_n(\tau_2)}, \qquad \frac{z_n'(\tau_2)}{z_n(\tau_2)}$$

But these functions have been calculated: they are the functions $f_1(n)$ $f_3(n)$ (see subsection 3) when $\lambda = 1/2\,\pi$. By virtue of this we can use the analysis already presented and write down the expansion of these functions as regards their principal parts:

$$\frac{1}{z_n(\tau_2)} = \frac{1}{z_0(\tau_2)} + 2\sum_{j=1}^{\infty} \frac{\xi_j}{m_j} - 2\sum_{j=1}^{\infty} \frac{\xi_j m_j}{n^2 + m_j^2}$$

$$\frac{z_n'(\tau_2)}{z_n(\tau^2)} = \frac{z_0'(\tau^2)}{z_0(\tau^2)} + 2\sum_{j=1}^{\infty} \frac{\zeta_j}{m_j} - 2\sum_{j=1}^{\infty} \frac{\zeta_j m_j}{n^2 + m_j^2}$$

where

$$i\xi_j = \left[\frac{1}{\partial z_n(\tau_2)/\partial n}\right]_{n=im_j}, \qquad i\zeta_j = \left[\frac{z_n'(\tau_2)}{\partial z_n(\tau_2)/\partial n}\right]_{n=im_j} \qquad (2.15)$$

Let us substitute these expansions in formulas (2.6) and (2.9), obtaining

$$OC = \frac{16Q\tau_1}{\sqrt{2\alpha\tau_1}(1-\tau_1)^\beta}\times$$

$$\times\left\{\left[\frac{1}{4z_0(\tau_0)} + \frac{1}{2}\sum_{j=1}^{\infty}\frac{\xi_j}{m_j}\frac{1}{1+4m_j^2}\right]\sin^2\frac{1}{2}\theta_0 + \sum_{j=1}^{\infty}\frac{\xi_j}{1+4m_j^2}\frac{\operatorname{sh}^2 m_j\theta_0}{\operatorname{sh}\pi m_j}\right\} \quad (2.16)$$

$$2l = \frac{32Q}{\sqrt{2\alpha\tau_2}(1-\tau_2)^\beta}\times$$

$$\times\left\{\left[\frac{1}{8} + \frac{\tau_2 z_0'(\tau_2)}{4z_0(\tau_2)} + \frac{\tau_2}{2}\sum_{j=1}^{\infty}\frac{1}{m_j}\frac{\zeta_j}{1+4m_j^2}\right]\sin^2\frac{1}{2}\theta_0 + \tau_2\sum_{j=1}^{\infty}\frac{\zeta_j}{1+4m_j^2}\frac{\operatorname{sh}^2 m_j\theta_0}{\operatorname{sh}\pi m_j}\right\} \quad (2.17)$$

If the velocities V_1 and V_2 are close to one another, then these two last formulas are appreciably simplified. These simplifications arise by virtue of the fact that, for values τ_1 and τ_2 differing only slightly from one another, the numbers m_j are determined by the following formula (see subsection 5):

$$m_j = \frac{\pi}{\nu'}j \qquad (j = \pm 1, \pm 2, \pm 3, \ldots)$$

which shows that all the m_j, starting from m_1 and m_{-1}, are very large for small ν'. Hence it follows that the two last infinite sums on the right-hand sides of formulas (2.16) and (2.17) can be neglected. However, by virtue of the formulas at the end of subsection 5, the sums

$$\sum_{j=1}^{\infty}\frac{\xi_j}{m_j}\frac{1}{1+4m_j^2}, \qquad \sum_{j=1}^{\infty}\frac{\zeta_j}{m_j}\frac{1}{1+4m_j^2}$$

take the following values respectively:

$$-\frac{1}{48}\left[\frac{\sqrt{1-(2\beta+1)\tau_2}}{\tau_2\sqrt{1-\tau_2}}\right]^2(\tau_2-\tau_1), \qquad \frac{1}{24}\left[\frac{\sqrt{1-(2\beta+1)\tau_2}}{\tau_2\sqrt{1-\tau_2}}\right]^2(\tau_2-\tau_1)$$

if we retain only the first terms in the expansions in powers of the small difference $\tau_2 - \tau_1$.

Hence, from formula (2.17) we obtain

$$l = \frac{4Q\tau_2}{(1-\tau_2)^\beta\sqrt{2\alpha\tau_2}}\frac{\sin^2 {}^1/_2\theta_0}{\tau_2-\tau_1} \quad (2.18)$$

Let us now take formula (2.11) and substitute therein for $\sin^2 1/2\ \theta_0$ its value from the last formula; we then obtain an expression for the pressure of the stream on the plate for values of τ_1 and τ_2 which differ only slightly:

$$R = \frac{2(\beta + 1)}{1 - \tau_2} l(\tau_2 - \tau_1) p_1$$

We observe that, by retaining only the first power of the difference $\tau_2 - \tau_1$, we cannot distinguish between the lengths OC and l.

Formula (2.18) can be used to determine the angle θ_0 of separation of the jets to infinity.

III. Flow of a gas out of a vessel

12. The partial differential equation, which is satisfied by the stream function $\psi(\theta, \tau)$ in plane-parallel potential motion of a gas, has the following form:

$$\frac{\partial}{\partial \tau} \left\{ \frac{2\tau}{(1-\tau)^\beta} \frac{\partial \psi}{\partial \tau} \right\} + \frac{1 - (2\beta + 1)\tau}{2\tau(1-\tau)^{\beta+1}} \frac{\partial^2 \psi}{\partial \theta^2} = 0 \qquad (3.1)$$

In order to construct solutions of problems on jet flows of a gas, Chaplygin found a number of particular solutions of equation (3.1). However, for the solution of the problems we mean to consider in this section, it is necessary to find other particular solutions of the same equation (3.1).

We will seek particular solutions of equation (3.1) expressible in the form of a product of two functions $\Theta(\theta)$ and $T(\tau)$, each depending upon only one argument. Substituting the product $\Theta(\theta)\, T(\tau)$ in place of the function ψ in equation (3.1) gives the following result:

$$\frac{1}{T} \frac{d}{d\tau} \left\{ \frac{2\tau}{(1-\tau)^\beta} \frac{dT}{d\tau} \right\} : \frac{1 - (2\beta + 1)\tau}{2\tau(1-\tau)^{\beta+1}} = -\frac{1}{\Theta} \frac{d^2\Theta}{d\theta^2}$$

Let us equate the common value of the right and left-hand sides of this equation to a certain negative number $-n^2$; then, in order to determine the unknown functions Θ and T, we get the following equations:

$$\frac{d^2\Theta}{d\theta^2} - n^2\Theta = 0, \qquad \frac{d}{d\tau} \left\{ \frac{2\tau}{(1-\tau)^\beta} \frac{dT}{d\tau} \right\} + n^2 \frac{1 - (2\beta + 1)\tau}{2\tau(1-\tau)^{\beta+1}} T = 0 \qquad (3.2)$$

The first equation can be integrated in hyperbolic functions, and its general integral can be written thus:

$$\Theta = A \operatorname{ch} n\theta + B \operatorname{sh} n\theta \qquad (3.3)$$

Integration of the second equation can be achieved in terms of the hypergeometric series. Let us set

$$T = \tau^{1/2 n i} S(\tau)$$

then for the determination of the function $S(\tau)$ we get the following equation, due to Gauss:

$$\tau(1-\tau)\frac{d^2S}{d\tau^2}+[(1+ni)+(\beta-1-ni)\tau]\frac{dS}{d\tau}+\frac{1}{2}i\beta n(1+ni)S=0$$

The solution of this equation is the hypergeometric series $F(a, b, c; \tau)$ with the parameters a, b, c defined by the formulas

$$a+b=ni-\beta, \quad ab=-\frac{1}{2}i\beta n(1+ni), \quad c=1+ni \tag{3.4}$$

Accordingly, the second equation (3.3) has the particular solution

$$T=\left(\frac{\tau}{\tau_2}\right)^{1/2ni}F(a, b, c; \tau)$$

Here τ_2 is an arbitrary constant number.

Proceeding in a similar manner, we find that equation (3.2) also has a particular solution of the following form:

$$T=\left(\frac{\tau_2}{\tau}\right)^{1/2ni}F(\bar{a}, \bar{b}, \bar{c}; \tau)$$

where the parameters $\bar{a}$, $\bar{b}$, $\bar{c}$ of the new hypergeometric series are determined by the equations

$$\bar{a}+\bar{b}=-ni-\beta, \quad ab=\frac{1}{2}i\beta n(1-ni), \quad \bar{c}=1-ni \tag{3.5}$$

By means of the particular solutions T and $\overline{T}$, let us now form new particular solutions T' and T'' of equation (3.2), setting

$$T'=\frac{1}{2}(T+\overline{T}), \qquad T''=\frac{1}{2i}(T-\overline{T})$$

These particular solutions can be expressed in the following form:

$$T'=M(\tau, n)\cos\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)-N(\tau, n)\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right) \tag{3.6}$$
$$T''=M(\tau, n)\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)+N(\tau, n)\cos\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)$$

where the functions $M(\tau, n)$ and $N(\tau, n)$ are two functions of the variables τ and the parameter n, expressed by the following series:

$$M(\tau; n)=1+p_1(n)\tau+p_2(n)\tau^2+\ldots$$
$$N(\tau, n)=[q_1(n)\tau+q_2(n)\tau^2+\ldots]n \tag{3.7}$$

where p_1, p_2, ..., q_1, q_2, ... are rational functions of the parameter n, containing in their expressions only even powers of this parameter.

Let us write down the expressions of the first few of these functions.

We have

$$p_1(n) = 0, \qquad q_1(n) = -\tfrac{1}{2}\beta$$

$$p_2(n) = \frac{n^2(1 - {}^1/_2\beta n^2)}{2(n^2+4)}\beta, \qquad q_2(n) = \frac{2(\beta+1) - n^2(1+{}^1/_2\beta)}{2(n^2+4)}\beta$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

13. Let us consider a vessel bounded by two parallel walls, extending in one direction to infinity and supplied with a nozzle, formed by two small and equal straight segments of wall, inclined to the centre line of the vessel and joined to the free ends of the parallel walls mentioned above (Fig. 6). From this vessel, gas issues under pressure in the form of a jet into free space. Our problem consists of constructing the stream function of this gas flow*.

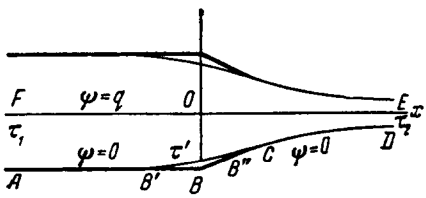

Fig. 6.

Let us assume that the value of the stream function ψ along the line of symmetry FE of the stream is $q > 0$, and along the compound boundary $ABCD$, including the free surface of the jet CD, is zero. Let us further assume that in the distant part of the vessel, from which the gas is coming, the value of the variable τ is given and is equal to τ_1; let us assume, moreover, that at the points of the free surface of the jet the variable τ has the value $\tau_2 < \tau_1$.

At the point B the velocity of the gas is equal to zero, and this circumstance introduces a well-known complication into the given problem. In order to avoid this difficulty we will first solve a somewhat altered problem, obtained by replacing the critical point of zero velocity by a region of stagnant gas $B'BB''$, along the curvilinear boundary $B'B''$ of which the variable τ has the small value τ'.

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* S.V. Fal'kovich solved the problem considered here by dividing the region of flow into two parts; in one part, containing the jet and the point of zero velocity, the stream function is given by a series of functions $z_n(\tau)$; in the other part, containing the remainder of the pipe, the stream function is expressed as a series, the general term of which contains the second solution of the hypergeometric equation [5].

Let us now consider the range of variation of the variables in the plane of θ and τ corresponding to the gas flow under consideration (Fig. 7). This region is the rectangle $AB'B''CDEF$,

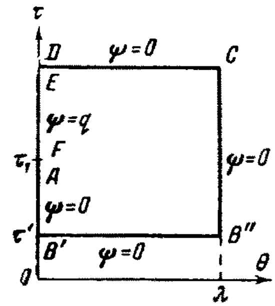

Fig. 7.

bounded by the straight lines $\tau = \tau'$, $\theta = \lambda$, $\tau = \tau_2$, $\theta = 0$; here λ is the angle of inclination of the wall segment BC to the axis OX. The required function $\psi(\theta, \tau)$ has to satisfy equation (3.1) and the following boundary conditions on the sides of the rectangle under consideration

$\psi = 0$ when $\tau = \tau'$, $\psi = 0$ when $\theta = \lambda$, $\psi = 0$ when $\tau = \tau_2$

$\psi = 0$ when $\theta = 0$ and $\tau' < \tau < \tau_1$, $\psi = q$ when $\theta = 0$ and $\tau_1 < \tau < \tau_2$

In order to find the function $\psi(\theta, \tau)$, let us consider this particular solution of equation (3.1):

$$\psi_n = A_n T_n(\tau)\,\mathrm{sh}\, n(\lambda - \theta)$$

If we subject the function $T_n(\tau)$ to the conditions

$$T_n(\tau') = 0. \qquad T_n(\tau_2) = 0 \tag{3.8}$$

then the function ψ_n will satisfy all the boundary conditions imposed upon the function $\psi(\theta, \tau)$, apart from the condition on the side $\theta = 0$. In order to satisfy this condition too, let us form a series with indeterminate coefficients A_n:

$$\psi(\theta, \tau) = \sum_n A_n T_n(\tau)\,\mathrm{sh}\, n(\lambda - \theta) \tag{3.9}$$

summing over all the fundamental numbers n of the equation (3.2) which satisfy the boundary conditions (3.8).

Two fundamental functions $T_n(\tau)$ and $T_m(\tau)$ of the equation, corresponding to two different fundamental numbers n and m, satisfy the integral relation:

$$\int_{\tau'}^{\tau_2} \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\, T_n(\tau)\, T_m(\tau)\, d\tau = 0$$

Let us assume, in addition, that

$$\int_{\tau'}^{\tau_2} \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}[T_n(\tau)]^2 d\tau = 1 \tag{3.10}$$

On the basis of these conditions of orthogonality and generalised normality of the functions $T_n(\tau)$, we can determine the coefficients A_n of the series (3.9). Let us set $\theta = 0$ in this series and, multiplying both sides by

$$\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}$$

let us integrate the result with respect to τ from τ' to τ_2. Taking into account the boundary conditions imposed upon the function $\psi(\theta, \tau)$, we find that

$$A_n = \frac{q}{\operatorname{sh} n\lambda} \int_{\tau_1}^{\tau_2} \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}} T_n(\tau)\, d\tau$$

Making use of the differential equation (3.2), we can effect the quadrature and in this way obtain for A_n the following expression:

$$A_n = \frac{q}{n^2 \operatorname{sh} n\lambda}\left[\frac{2\tau}{(1-\tau)^{\beta}}\frac{dT_n}{d\tau}\right]_{\tau_2}^{\tau_1}$$

Accordingly, the series which is the solution of this preliminary problem can be written thus:

$$\psi(\theta, \tau) = q \sum_n \left[\frac{2\tau}{(1-\tau)^{\beta}}\frac{dT_n}{d\tau}\right]_{\tau_2}^{\tau_1} \frac{\operatorname{sh} n(\lambda-\theta)}{n^2 \operatorname{sh} n\lambda} T_n(\tau) \tag{3.11}$$

Using this series, we can calculate all the elements determining the motion of the gas.

14. In order to solve the problem originally formulated, when instead of the stagnation region $BB'B''$ there is a single stagnation point B, we have to let the number τ' in formula (3.11) tend to zero. To achieve this passage to the limit, we have to record certain intermediate propositions.

Let us first take equation (3.2) and transform it to a new form; in place of the independent variable τ and the function $T(\tau)$ let us introduce the new independent variable z and the new unknown function $u(z)$, by setting

$$u(z) = T(\tau)\sqrt{E(\tau)}, \quad z = \int_{\tau}^{\tau_2} \frac{d\tau}{2\tau}\sqrt{\frac{1-(2\beta+1)\tau}{1-\tau}} \qquad \left(E(\tau) = \sqrt{\frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}}}\right)$$

We then obtain the following differential equation:

$$\frac{d^2u}{dz^2}+\left[n^2+\frac{\beta(2\beta+1)\tau^2}{1-\tau}\;\frac{4-2(\beta+2)\tau-\beta(2\beta+1)\tau^2}{[1-(2\beta+1)\tau]^3}\right]u=0 \qquad (3.12)$$

We observe that for subsonic motion of the gas the numerator of the second term in the square brackets is positive.

In accordance with the boundary conditions (3.8) we have to consider those integrals of this equation which satisfy the following conditions:

$$u_n(0)=0, \qquad u_n(z')=0 \qquad (3.13)$$

where the zero value of the variable z corresponds to r_2, and the value z' corresponds to the value r' of the variable r.

From the formula determining z from r it follows that, as r' tends to zero, the quantity z' will tend to infinity, and accordingly the second boundary condition (3.13) has to be satisfied for very large values of the independent variable z. Using the theory of asymptotic representations of integrals of linear differential equations, it is possible to give approximate values for the fundamental numbers of equation (3.12), corresponding to large values of the quantity z'; we have

$$n_j=\frac{\pi j}{z'} \qquad (j=1,2,3,\ldots) \qquad (3.14)$$

Hence we see that the difference, between two successive fundamental numbers n appearing in the series (3.11), is equal to π/z' and consequently tends to zero as z' tends to infinity. By virtue of this fact we can assume that, as $z'\to\infty$, the sum (3.11) will tend to a certain definite integral. In order to construct this integral we shall need to consider in somewhat greater detail the general term of series (3.11).

The function $T_n(r)$, appearing in the general term of this series, can be represented in terms of particular solutions (3.6) of the equation (3.3) in the following way:

$$T_n(\tau)=C_n\,[T_n'(\tau_2)\,T_n''(\tau)-T_n''(\tau_2)\,T_n'(\tau)]$$

where the coefficient C_n has to be determined from the condition (3.10). Let us present this condition in a new form, introducing in place of the variable of integration r the new variable σ, by putting

$$\sigma=\frac{1}{2}\ln\frac{\tau_2}{\tau}, \qquad \sigma'=\frac{1}{2}\ln\frac{\tau_2}{\tau'}$$

We obtain

$$\int_0^{\sigma'}\frac{1-\tau_2(2\beta+1)\,e^{-2\sigma}}{(1-\tau_2e^{-2\sigma})^{\beta+1}}\,[T_n(\sigma)]^2\,d\sigma=1 \qquad (3.15)$$

We observe that the expressions for the functions $T_n'(\tau)$ and $T_n''(\tau)$, in terms of the new variable σ, are

$$T' = [1 + p_1(n)\tau_2 e^{-2\sigma} + \ldots]\cos n\sigma - n[q_1(n)\tau_2 e^{-2\sigma} + \ldots]\sin n\sigma$$
$$T'' = [1 + p_1(n)\tau_2 e^{-2\sigma} + \ldots]\sin n\sigma + n[q_1(n)\tau_2 e^{-2\sigma} + \ldots]\cos n\sigma$$

Hence we obtain

$$T_n(\sigma) = C_n\{[T_n'(\tau_2) + a_1 e^{-2\sigma} + a_2 e^{-4\sigma} + \ldots]\sin n\sigma - \\ - [T_n''(\tau_2) + b_1 e^{-2\sigma} + b_2 e^{-4\sigma} + \ldots]\cos n\sigma\}$$

where $a_1, a_2, \ldots, b_1, b_2, \ldots$ are completely determined coefficients depending on the number n. We can now, moreover, write,

$$[T_n(\sigma)]^2 = C_n^2\Big\{\tfrac{1}{2}[M^2(\tau_2, n) + N^2(\tau_2, n)] + k_1 e^{-2\sigma} + k_2 e^{-4\sigma} + \ldots + \\ + (l_0 + l_1 e^{-2\sigma} + \ldots)\cos 2n\sigma + (m_0 + m_1 e^{-2\sigma} + \ldots)\sin 2n\sigma\Big\} \tag{3.16}$$

where $k_1, k_2, \ldots, l_0, l_1, l_2, \ldots, m_0, m_1, m_2, \ldots$ are certain numbers depending on the parameter n.

We can also write down the following expansion:

$$\frac{1 - \tau_2(2\beta + 1)e^{-2\sigma}}{1 - \tau_2 e^{-2\sigma})^{\beta+1}} = 1 + A_1 e^{-2\sigma} + A_2 e^{-4\sigma} + \ldots$$

Now, the condition (3.15) can be written thus:

$$\frac{1}{C_n^2} = \tfrac{1}{2}[M^2(\tau_2, n) + N^2(\tau_2, n)]\int_0^{\sigma'} d\sigma + \sum_{j=1}^{\infty} B_j \int_0^{\sigma'} e^{-2j\sigma} d\sigma + \\ + \sum_{j=0}^{\infty} C_j \int_0^{\sigma'} e^{-2j\sigma}\cos 2n\sigma\, d\sigma + \sum_{j=0}^{\infty} D_j \int_0^{\sigma'} e^{-2j\sigma}\sin 2n\sigma\, d\sigma \tag{3.17}$$

where B_j, C_j, D_j are completely determined numbers. We have

$$\int_0^{\sigma'} d\sigma = \sigma', \qquad \int_0^{\sigma'} e^{-2j\sigma}\, d\sigma = \frac{1}{2j}(1 - e^{-2j\sigma'})$$

$$\int_0^{\sigma'} e^{-2j\sigma}\cos 2n\sigma\, d\sigma = \frac{e^{-2j\sigma'}(n\sin 2n\sigma' - j\cos 2n\sigma') + j}{2(j^2 + n^2)}$$

$$\int_0^{\sigma'} e^{-2j\sigma}\sin 2n\sigma\, d\sigma = \frac{n - e^{-2j\sigma'}(j\sin 2n\sigma' + n\cos 2n\sigma')}{2(j^2 + n^2)}$$

From these formulas it follows that the relation (3.17), from which C_n is to be determined, can be rewritten thus:

$$\frac{1}{C_n^2} = \frac{1}{2}\left[M^2(\tau_2,\ n) + N^2(\tau_2,\ n)\right]\sigma'\{1 + L(\sigma')\}$$

where the functions $L(\sigma')$ are certain functions of σ' which tend to zero as σ' tends to infinity.

Accordingly, the functions $T_n(r)$ appearing in the series (3.11) can be written thus:

$$T_n(\tau) = \frac{\vartheta(\tau, n)}{\sqrt{M^2(\tau_2, n) + N^2(\tau_2, n)}}\sqrt{\frac{2}{\sigma'}}\frac{1}{\sqrt{1 + L(\sigma')}}$$

where

$$\vartheta(\tau,\ n) = T_n'(\tau_2)\,T_n''(\tau) - T_n''(\tau_2)\,T_n'(\tau)$$

Now the series (3.11), which was the solution of the preliminary problem, can be written so:

$$\psi(\theta,\ \tau) = \frac{2q}{\sigma'}\sum_n\left\{\frac{2\tau_1}{(1-\tau_1)^\beta}\frac{d\vartheta(\tau_1,\ n)}{d\tau_1} - \frac{2\tau_2}{(1-\tau_2)^\beta}\frac{d\vartheta(\tau_2,\ n)}{d\tau_2}\right\}\times$$
$$\times\frac{\operatorname{sh} n(\lambda-\theta)}{n^2 \operatorname{sh} n\lambda}\frac{\vartheta(\tau,\ n)}{M^2(\tau_2, n) + N^2(\tau_2, n)}\frac{1}{1 + L(\sigma')} \tag{3.18}$$

Here it is necessary to make one important remark concerning the notation: the function $\theta(r,\ n)$ depends on the quantities r_1 and r_2, but when we write $d\theta(r_1,\ n)/dr_1$ and $d\theta(r_2,\ n)/dr_2$ we mean to signify the values of the derivative $d\theta(r,\ n)/dr$ evaluated at r_1 and r_2 respectively; accordingly, r_1 and r_2, which appear in the function $\theta(r,\ n)$ as parameters, are not subject to differentiation.

Between the numbers r', z', σ' there exist the following relations

$$z' = \int_{\tau'}^{\tau_2}\frac{d\tau}{2\tau}\sqrt{\frac{1-(2\beta+1)\tau}{1-\tau}}, \qquad \sigma' = \frac{1}{2}\ln\frac{\tau_2}{\tau'}$$

From the first relation we obtain

$$z' = \frac{1}{2}\ln\frac{\tau_2}{\tau'} - \frac{1}{2}(\beta+1)(\tau_2 - \tau') + \ldots$$

Hence it follows that for small r' we can take $z' = \sigma'$.

By virtue of this fact, the formula (3.18) for the stream function can be rewritten in a new form; taking into account the connection (3.14) between n_j and z', we have

$$\Delta n = \frac{\pi}{z'} = \frac{\pi}{\sigma'}$$

Accordingly, we obtain

$$\psi(\theta,\ \tau)=\frac{2q}{\pi}\sum_{j=1}^{\infty}\left\{\frac{2\tau_1}{(1-\tau_1)^{\beta}}\frac{d\vartheta(\tau_1,\ n)}{d\tau_1}-\frac{2\tau_2}{(1-\tau_2)^{\beta}}\frac{d\vartheta(\tau_2,\ n)}{d\tau_2}\right\}\frac{\operatorname{sh} n(\lambda-\theta)}{n^2 \operatorname{sh} n\lambda}\times$$
$$\times\frac{\vartheta(\tau,\ n)}{M^2(\tau_2,\ n)+N^2(\tau_2,\ n)}\frac{\Delta n}{1+L(\sigma')}$$

Let us now pass to the limit, when the number r' tends to zero. In the limit we find the following expression for $\psi(\theta,\ r)$:

$$\psi(\theta,\ \tau)=\frac{2q}{\pi}\int_0^{\infty}\left\{\frac{2\tau_1}{(1-\tau_1)^{\beta}}\frac{d\vartheta(\tau_1,\ n)}{d\tau_1}-\frac{2\tau_2}{(1-\tau_2)^{\beta}}\frac{d\vartheta(\tau_2,\ n)}{d\tau_2}\right\}\times$$
$$\times\frac{\operatorname{sh} n(\lambda-\theta)}{n^2 \operatorname{sh} n\lambda}\frac{\vartheta(\tau,\ n)\,dn}{M^2(\tau_2,\ n)+N^2(\tau_2,\ n)} \tag{3.19}$$

By means of this formula we can find the contraction of the jet and the pressure of the gas stream on the sides of the nozzle.

Let us turn to Fig. 6 and find the connection between the ordinates of the points C and D; let us denote these ordinates by $-H$ and $-h$, respectively.

We have the following general formula of the Chaplygin method:

$$dz=\frac{e^{i\theta}}{\sqrt{2\alpha\tau}}\left(d\varphi+i\frac{\rho_0}{\rho}d\psi\right)$$

where ρ_0 is the density at that point of the gas stream where the velocity of the gas is equal to zero.

Let us apply this formula to the arc CD of the streamline $\psi = 0$; we obtain

$$dz=\frac{e^{i\theta}}{\sqrt{2\alpha\tau_2}}\frac{2\tau_2}{(1-\tau_2)^{\beta}}\left(\frac{\partial\psi}{\partial\tau}\right)_{\tau=\tau_2}d\theta$$

Hence

$$dy=\frac{1}{\sqrt{2\alpha\tau_2}}\frac{2\tau_2}{(1-\tau_2)^{\beta}}\left(\frac{\partial\psi}{\partial\tau}\right)_{\tau=\tau_2}\sin\theta\,d\theta$$

and consequently

$$H-h=-\frac{1}{\sqrt{2\alpha\tau_2}}\frac{2\tau_2}{(1-\tau_2)^{\beta}}\int_0^{\lambda}\left(\frac{\partial\psi}{\partial\tau}\right)_{\tau=\tau_2}\sin\theta\,d\theta$$

Substituting in the right-hand side of this formula for the function $\psi(\theta,\ r)$ its expression in the form of the integral (3.19) and carrying out the calculation, we arrive at the following result:

$$H-h=-\frac{2q}{\pi\alpha}\frac{\sqrt{2\alpha\tau_2}}{(1-\tau_2)^\beta}\int\limits_0^\infty\left\{\frac{2\tau_1}{(1-\tau_1)^\beta}\frac{d\vartheta(\tau_1,n)}{d\tau_1}-\frac{2\tau_2}{(1-\tau_2)^\beta}\frac{d\vartheta(\tau_2,n)}{d\tau_2}\right\}\times$$
$$\times\frac{\operatorname{sh}n\lambda-n\sin n\lambda}{n^2(1+n^2)\operatorname{sh}n\lambda}\frac{1}{M^2(\tau_2,n)+N^2(\tau_2,n)}\left(\frac{d\vartheta}{d\tau}\right)_{\tau_2}dn \qquad (3.20)$$

15. Let us apply these formulas to the case of the flow of an incompressible liquid.

In this particular case $\beta = 0$ and equation (3.2) has the following integrals:

$$T'=\cos\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right),\qquad T''=\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)$$

and therefore

$$\vartheta(\tau,n)=\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)$$

$$\frac{2\tau_1}{(1-\tau_1)^\beta}\frac{d\vartheta(\tau_1,n)}{d\tau_1}-\frac{2\tau_2}{(1-\tau_2)^\beta}\frac{d\vartheta(\tau_2,n)}{d\tau_2}=n\left[1-\cos\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau_1}\right)\right]$$

Let us now evaluate formula (3.19); we obtain

$$\psi(\theta,\tau)=\frac{2q}{\pi}\int\limits_0^\infty\frac{1-\cos\mu n}{n}\frac{\operatorname{sh}n(\lambda-\theta)}{\operatorname{sh}n\lambda}\sin\left(\frac{1}{2}n\ln\frac{\tau_2}{\tau}\right)dn \qquad (3.21)$$

where

$$\mu=\frac{1}{2}\ln\frac{\tau_2}{\tau_1}=\ln\frac{V_2}{V_1}$$

and V_1 is the velocity of the liquid in the remote parts of the vessel, whilst V_2 is the velocity of the liquid in the jet.

From formula (3.21) we easily find the expression for the complex flow function $w(\sigma)$ of the complex variable σ, introduced by the formula

$$\sigma=\ln\frac{V_2}{V}+i(\theta-\lambda)$$

We obtain

$$w(\sigma)=\frac{2q}{\pi}\int\limits_0^\infty\frac{1-\cos\mu n}{n}\frac{\cos\sigma n}{\operatorname{sh}\lambda n}dn$$

Hence we find, on carrying out the quadrature, that

$$\frac{dw}{d\sigma}=-\frac{q}{\lambda}\left\{\operatorname{th}\frac{\pi\sigma}{2\lambda}-\frac{1}{2}\operatorname{sh}\frac{\pi\sigma}{\lambda}\operatorname{sch}\frac{\pi(\mu+\sigma)}{2\lambda}\operatorname{sch}\frac{\pi(\mu-\sigma)}{2\lambda}\right\} \qquad (3.22)$$

From these formulas we find the relation between H and h, thus:

$$H-h=-\frac{i}{\sqrt{2a\tau_2}}\int_0^{\lambda}\left(\frac{dw}{d\sigma}\right)_{\sigma=i(\theta-\lambda)}\sin\theta\,d\theta$$

Substituting herein the expression (3.22) for $dw/d\sigma$, and taking $\sigma = i(\theta - \lambda)$, we obtain

$$\frac{H-h}{h}=\frac{1}{\lambda}\int_0^{\lambda}\sin\theta\operatorname{ctg}\frac{\pi\theta}{2\lambda}\left(1+\frac{\sin^2(\pi\theta/2\lambda)}{\operatorname{sh}^2(\pi\mu/2\lambda)}\right)^{-1}d\theta$$

This integral can be calculated when $\lambda = 1/2\,\pi$, and we obtain

$$\frac{H-h}{h}=\frac{2}{\pi}\operatorname{sh}\mu\operatorname{arc\,tg}\left(\frac{1}{\operatorname{sh}\mu}\right)$$

If in place of μ we introduce the new parameter γ, by setting

$$\operatorname{tg}\gamma=e^{-\mu}=\frac{V_1}{V_2}$$

then we put the foregoing formulas into the following form:

$$\frac{H}{h}=1+\frac{4}{\pi}\frac{\gamma}{\operatorname{th}2\gamma}$$

Let us introduce here in place of h the quantity $2L$, equal to the width of the vessel; we then obtain the formula given by Zhukovskii:

$$\frac{H}{L}=\operatorname{th}\gamma\left(1+\frac{4}{\pi}\frac{\gamma}{\operatorname{tg}2\gamma}\right)$$

From this formula we can, given H and L, determine the number γ, and hence the velocity of flow in the jet, if the velocity of the liquid in the remote parts of the vessel is known.

16. The problem of the flow of gas out of a vessel, which was solved in the foregoing subsection, is equivalent to the problem of the motion of gas through a grating consisting of bars of equal size. Let us suppose that $\lambda = 1/2\,\pi$, then we have a segment of the flow past a regular grating with jet formation (Fig. 8).

Along a bar of the grating we have

$$dy=-\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta}\frac{d\tau}{\sqrt{2a\tau}}$$

In order to determine the pressure P, acting on each bar of the grating, we use the formula:

$$p=p_0(1-\tau)^{\beta+1}$$

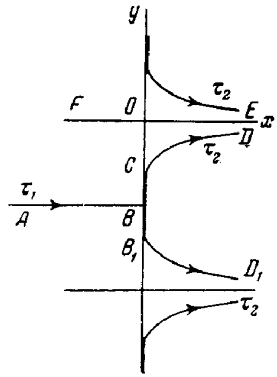

Fig. 8.

Hence we obtain for P the following formula:

$$P = -2p_0 \int_0^{\tau_2} \frac{1-(2\beta+1)\tau}{2\tau\sqrt{2\alpha\tau}} \left(\frac{\partial\psi}{\partial\theta}\right)_{\theta=\frac{1}{2}\pi} d\tau$$

Substituting here in place of $\psi(\theta, \tau)$ the expression (3.19), we obtain

$$P = \frac{4p_0 q}{\pi} \lim_{\tau'=0} \int_0^\infty \left\{\frac{2\tau_1}{(1-\tau_1)^\beta} \frac{d\vartheta(\tau_1, n)}{d\tau_1} - \frac{2\tau_2}{(1-\tau_2)^\beta} \frac{d\vartheta(\tau_2, n)}{d\tau_2}\right\} \times$$

$$\times \frac{1}{n \operatorname{sh} {}^1/_2\pi n} \frac{dn}{M^2(\tau_2, n) + N^2(\tau_2, n)} \int_{\tau'}^{\tau_2} \frac{1-(2\beta+1)\tau}{2\tau\sqrt{2\alpha\tau}} \vartheta(\tau, n)\, d\tau$$

But

$$\int_{\tau'}^{\tau_2} \frac{1-(2\beta+1)\tau}{2\tau\sqrt{2\alpha\tau}} \vartheta(\tau, n)\, d\tau = -\frac{1}{(n^2+1)\sqrt{2\alpha}} \left\{2\sqrt{\tau_2}(1-\tau_2)\frac{d\vartheta(\tau_2, n)}{d\tau_2} - \right.$$

$$\left. -2\sqrt{\tau'}(1-\tau')\frac{d\vartheta(\tau', n)}{d\tau'} - \frac{1+(2\beta+1)\tau'}{\sqrt{\tau'}} \vartheta(\tau', n)\right\}$$

Consequently,

$$P = -\frac{8p_0 q}{\pi} \frac{\tau_2(1-\tau_2)}{\sqrt{2\alpha\tau_2}} \int_0^\infty \{\Gamma(\tau_1, \tau_2; n)\} \frac{d\vartheta(\tau_2, n)}{d\tau_2} \times$$

$$\times \frac{1}{n(n^2+1)\operatorname{sh} {}^1/_2\pi n} \frac{dn}{M^2(\tau_2, n) + N^2(\tau_2, n)} +$$

$$+ \frac{4p_0 q}{\pi\sqrt{2\alpha}} \lim_{\tau'=0} \int_0^\infty \left\{2\sqrt{\tau'}(1-\tau')\frac{d\vartheta(\tau', n)}{d\tau'} + \frac{1+(2\beta+1)\tau'}{\sqrt{\tau'}}\vartheta(\tau', n)\right\} \times$$

$$\times \Gamma(\tau_1, \tau_2, n) \frac{1}{n(n^2+1)\operatorname{sh} {}^1/_2\pi n} \frac{dn}{M^2(\tau_2, n) + N^2(\tau_2, n)} \qquad (3.23)$$

where

$$\Gamma(\tau_1, \tau_2, n) = \frac{2\tau_1}{(1-\tau_1)^\beta}\frac{d\vartheta(\tau_1, n)}{d\tau_1} - \frac{2\tau_2}{(1-\tau_2)^\beta}\frac{d\vartheta(\tau_2, n)}{d\tau_2}$$

In equation (3.23) let us effect the passage to the limit; for small τ' we have

$$T' = \cos\left(\frac{1}{2} n \ln\frac{\tau_2}{\tau}\right), \qquad T'' = \sin\left(\frac{1}{2} n \ln\frac{\tau_2}{\tau}\right)$$

$$\vartheta(\tau', n) = \sin\left(\frac{1}{2} n \ln\frac{\tau_2}{\tau'}\right), \qquad \frac{d\vartheta(\tau', n)}{d\tau'} = -\frac{n}{2\tau'}\cos\left(\frac{1}{2} n \ln\frac{\tau_2}{\tau'}\right)$$

Hence the second term of formula (3.23) can be rewritten thus:

$$\frac{4p_0 q}{\pi\sqrt{2\alpha}} \lim_{\tau'=0} \int_0^\infty \frac{1-\cos({}^1\!/_2 n \ln \tau_2/\tau_1)}{(n^2+1)\,\mathrm{sh}\,{}^1\!/_2\,\pi n} \times$$

$$\times\left\{\frac{1+(2\beta+1)\tau'}{\sqrt{\tau'}}\sin\left(\frac{1}{2} n \ln\frac{\tau_2}{\tau'}\right) - n\frac{1-\tau'}{\sqrt{\tau'}}\cos\left(\frac{1}{2} n \ln\frac{\tau_2}{\tau'}\right)\right\} dn \qquad (3.24)$$

Let us calculate the definite integral

$$\frac{1+(2\beta+1)\tau'}{2\sqrt{\tau'}}\left[\pi(\mathrm{ch}\,\alpha - 1)\sqrt{\frac{\tau'}{\tau_2}} + \sum_{m=1}^{\infty}(-)^m \frac{\mathrm{ch}\,2m\alpha - 1}{m^2 - {}^1\!/_4}\left(\frac{\tau'}{\tau_2}\right)^m\right] + \frac{1-\tau'}{2\sqrt{\tau'}} \times$$

$$\times\left[\pi(\mathrm{ch}\,\alpha - 1)\sqrt{\frac{\tau'}{\tau_2}} + 2\sum_{m=1}^{\infty}(-)^m m\frac{\mathrm{ch}\,2m\alpha - 1}{m^2 - {}^1\!/_4}\left(\frac{\tau'}{\tau_2}\right)^m\right] \qquad \left(e^\alpha = \sqrt{\frac{\tau_2}{\tau_1}}\right)$$

Hence it is clear that the quantity (3.24) has the following simple value:

$$\frac{4p_0 q}{\pi\sqrt{2\alpha}}\ \frac{\pi}{2\sqrt{\tau_2}}\left(\sqrt[4]{\frac{\tau_2}{\tau_1}} - \sqrt[4]{\frac{\tau_1}{\tau_2}}\right)^2$$

Let us now turn to formula (3.23); we can now recast this formula in the following final form:

$$P = \frac{2p_0 q}{\sqrt{2\alpha\tau_2}}\left(\sqrt[4]{\frac{\tau_2}{\tau_1}} - \sqrt[4]{\frac{\tau_1}{\tau_2}}\right)^2 - \qquad (3.25)$$

$$-\frac{8p_0 q}{\pi}\frac{\tau_2(1-\tau_2)}{\sqrt{2\alpha\tau_2}}\int_0^\infty \Gamma(\tau_1, \tau_2, n)\frac{d\vartheta(\tau_2, n)}{d\tau_2}\frac{1}{n(n^2+1)\,\mathrm{sh}\,{}^1\!/_2\pi n}\frac{dn}{M^2(\tau_2, n) + N^2(\tau_2, n)}$$

For incompressible fluid this integral can be evaluated, and we then obtain the following formula for the pressure of the stream on the bar of the grating:

$$P = \frac{2q}{V_1 V_2}(V_1 - V_2)\left\{\frac{2p_2}{\pi V_2}(V_1 + V_2)\,\mathrm{arc\,tg}\,\frac{V_1}{V_2} + \frac{1}{2}\rho V_1(V_2 - V_1) - p_1\right\}$$

where p_1 and p_2 are the pressures in the stream in front of the grating and in the jet, respectively. If we deduct from this value the back-pressure, acting on the bar from the side of the stagnant fluid and equal to $2(L - H)p_2$, then we obtain Zhukovskii's formula:

$$P - 2(L - H)p_2 = 2\rho L V_1 V_2 \left(\frac{1}{\sin 2\gamma} - 1\right)$$

17. The problem of the flow of incompressible fluid out of a vessel into a pipe was first solved by Zhukovskii; we intend to solve this same problem for a gas. The configuration of the problem is illustrated in Fig. 9.

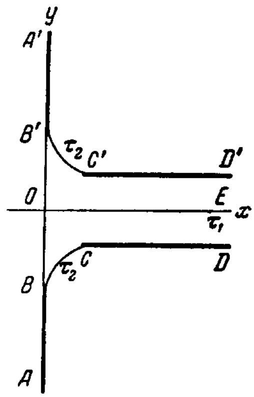

Fig. 9.

Gas from an infinite vessel, in the remote parts of which it is at rest, issues under pressure from the orifice BB', forms a short length of free jet bounded by BC and $B'C'$, and then enters an infinitely long pipe $CC'DD'$.

Let us assume that along the free surfaces Chaplygin's variable τ has the value τ_2, and in the remote parts of the pipe the variable τ is equal to $\tau_1 < \tau_2$. Let us assume that the value of stream function $\psi(\theta, \tau)$ along the line $ABCD$ is zero, and along the line CE is equal to $q > 0$.

Let us consider in place of the function $\psi(\theta, \tau)$ the new solution of equation (3.1) - the function $\Psi(\theta, \tau)$, connected with $\psi(\theta, \tau)$ by the formula

$$\Psi(\theta, \tau) = \psi(\theta, \tau) + \frac{2q}{\pi}\left(\theta - \frac{1}{2}\pi\right)$$

The new function $\Psi(\theta, \tau)$ will satisfy the boundary conditions:

$$\Psi = 0 \qquad \text{for } \theta = \tfrac{1}{2}\pi \ \& \ 0 < \tau < \tau_2$$

$$\Psi = \frac{2q}{\pi}\left(\theta - \frac{1}{2}\pi\right) \ \text{for } \tau = \tau_2 \ \& \ 0 < \theta < \tfrac{1}{2}\pi$$

$$\Psi(\theta, \tau) = \begin{cases} -q & \text{when } \tau_1 < \tau < \tau_2 \text{ and } \theta = 0 \\ 0 & \text{when } 0 < \tau < \tau_1 \text{ and } \theta = 0 \end{cases}$$

The required function $\Psi(\theta, \tau)$ can be represented in the form of a sum of two functions $\Psi_1(\theta, \tau)$ and $\Psi_2(\theta, \tau)$, defined by the formulas:

$$\Psi_1(\theta, \tau) = -\frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{z_n(\tau)}{z_n(\tau_2)}\frac{\sin 2n\theta}{n}$$

$$\Psi_2(\theta, \tau) = -\frac{2q}{\pi}\int_0^{\infty}\left\{\frac{2\tau_1}{(1-\tau_1)^{\beta}}\frac{d\vartheta(\tau_1, n)}{d\tau_1} - \right.$$

$$\left. - \frac{2\tau_2}{(1-\tau_2)^{\beta}}\frac{d\vartheta(\tau_2, n)}{d\tau_2}\right\}\frac{\operatorname{sh} n(^1/_2\pi - \theta)}{n^2 \operatorname{sh} {}^1/_2\pi n}\frac{\vartheta(\tau, n)\, dn}{M^2(\tau_2, n) + N^2(\tau_2, n)}$$

This last function can be obtained from formula (3.19) by replacing q by $-q$ and setting λ equal to $1/2\ \pi$.

18. Let us assume that gas issues from the open end of a pipe, at the far end of which the gas velocity is V_1; on issuing from the pipe, the gas impinges on a plane $C'BC$ and flows along it, forming free streamlines ED and $E'D'$ (Fig. 10). The velocity of flow along these lines is constant

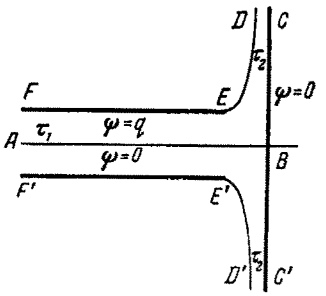

Fig. 10.

and equal to $V_2 > V_1$. The construction of the stream function for the gas flow can be effected in this case also by using formula (3.19).

Applying Chaplygin's function $z_n(\tau)$, as in the previous problem, we can write down an expression for the stream function in the following form:

$$\psi(\theta, \tau) = \frac{4q}{\pi}\sum_{n=1}^{\infty}{}' \frac{z_{2n}(\tau)}{z_{2n}(\tau_2)}\frac{\sin 2n\theta}{n} +$$

$$+ \frac{2q}{\pi}\int_0^{\infty}\left\{\frac{2\tau_1}{(1-\tau_1)^{\beta}}\frac{d\vartheta(\tau_1, n)}{d\tau_1} - \frac{2\tau_2}{(1-\tau_2)^{\beta}}\frac{d\vartheta(\tau_2, n)}{d\tau_2}\right\}\frac{\operatorname{sh} n(^1/_2\pi - \theta)}{n^2 \operatorname{sh} {}^1/_2\pi n}\frac{\vartheta(\tau, n)\, dn}{M^2(\tau_2, n) + N^2(\tau_2, n)}$$

The sign $'$ (prime) in the sum indicates that the index of summation n can assume only odd values.

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